PMATH 320: Euclidean Geometry

Spring 2026 course notes, University of Waterloo

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Contents

These notes are for the Spring 2026 offering of *PMATH 320: Euclidean Geometry*. They are not to be considered a comprehensive textbook alone. Together with your notes (mental and otherwise) from the lectures, they form the official reference for the course.

If you spot any typos or mistakes, please let me know!

These notes are being typeset in Typst, which is a new potential alternative to LaTeX. I am using these notes as an excuse to learn Typst. As a result, formatting may change repeatedly and erratically throughout the term. My apologies in advance.

Lecture 1 - 05/11

1. Geometry from Euclid and beyond

Euclidean geometry begins with, well, Euclid. In approximately 300BC, Euclid wrote *The Elements*, a truly transformational collection of thirteen books. It laid down the foundations for how we do mathematics today.

The first six books deal with plane geometry. Books VI to X cover what we would refer to today as Number Theory, although numbers and lengths are philosophically the same in these books. The collection of books finishes with books XI to XIII covering solid geometry.

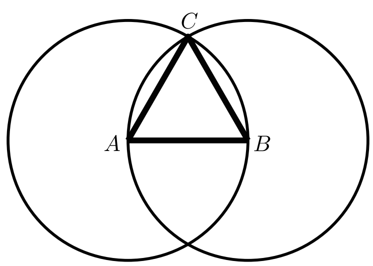
Book I begins with a list of 23 *definitions*, 5 *postulates*, and 5 *common notions*. They are listed in the appendix. These are things which we are to take for granted as true, and lay the foundations for what comes next. Upon these foundations, various propositions and constructions are built, gradually assembling a large collection of results, each building on what came before.

Let's see what this looked like. Here is the first proposition from Euclid's elements, rephrased in what I consider to be modern mathematical terminology.

Proposition 1. Given a line segment, there exists an equilateral triangle with that line segment as one of its edges.

Before we get into the proof of Proposition 1, it's worth noting that the definition of an equilateral triangle is in Definition 20 from Euclid's list of definitions; it's a triangle where all three sides have the same length.

Proof (of Proposition 1).



Let the line segment have endpoints A and B. Draw circles centered at A and B, each with radius AB (which we can do by Postulate 3.). Let C be a point on both circles. Draw AC and BC (which we can do by Postulate 1.). Then, by Definition 15 and Common Notion 1, we have

$$AB = AC = BC.$$

Therefore, the triangle ABC is equilateral (by Definition 20). ■

Notice that each step is very carefully justified by one of the definitions, postulates, or common notions. This is the style in which the books are written. That being said, by our modern standards of rigour, there are still holes. For example, how do we know that a point C exists? The picture certainly suggests it is so, but the existence of such a point cannot be deduced from the postulates, definitions, and common notions alone.

Lecture 2 - 05/13

The results in The Elements, or at least in Book I, tend to come in two flavours: constructions and propositions (a statement we can prove is either true or false). Proposition 1 is an example of the former, since it is shown how to construct an equilateral triangle given a line segment. Next is an example of the latter, which is Proposition 5 from the Book I.

Definition (Euclid's Definition 20). An *isosceles triangle* is a triangle with two sides of equal length.

Proposition 2. In an isosceles triangle, the angles at the base equal each other.

Instead of proving Proposition 2 like Euclid did, let's do it from a slightly more modern point of view, using symmetry!

Symmetry arguments like the one we are about to do are sometimes seen as not rigorous. When done correctly this is not at all true! Our argument will rely on some facts about reflections in the plane that are left as an exercise for you to prove. Proving these is what makes the argument that follows rigorous!

Fact 3. Let σ be the reflection of $\mathbb{R}^2$ across the line l. Then the following statements are true.

- $\sigma(x) = x$ for all $x \in l$.
- Angles are preserved by σ. That is, if θ is the angle between line segments AB and AC, then θ is the angle between line segments $\sigma(A)\sigma(B)$ and $\sigma(A)\sigma(C)$.
- Lengths are preserved by σ. That is, the distance between A and B is equal to the distance between $\sigma(A)$ and $\sigma(B)$.
- For all $x \in \mathbb{R}^2$, $\sigma \circ \sigma(x) = x$.

Proof. This is an exercise for you, and an excuse to dust off those linear algebra notes. ■

Let's use Fact 3 to prove Proposition 2.

Proof (of Proposition 2). Consider $\triangle ABC$ where $AB = AC$. Let l be a line bisecting $\angle BAC$, that is, the angle between l and AB is equal to the angle between l and AC. Let σ be the reflection across l. Since $A \in l$, $\sigma(A) = A$. Also, $\sigma(B)$ lies on the line passing through A and C as a result of σ preserving angles. Since σ preserves length, the distance between A and $\sigma(B)$ is equal to that between A and B, which by assumption, is equal to that between A and C. We are compelled to conclude that $\sigma(B) = C$. Applying σ to both sides gives

$$\sigma(C) = \sigma \circ \sigma(B) = B.$$

Putting all of this together we have $\sigma(A) = A$, $\sigma(B) = C$, and $\sigma(C) = B$. Since σ preserves angles, $\angle ABC = \angle BAC$ completing the proof. ■

Symmetry arguments can often feel sneaky or tricky. Go to a café or the pub with a friend and interrogate the proof!

Exercise. Prove that if a triangle has two equal angles, then it is an isosceles triangle.

Let's see another perspective with which we can do geometry: by coordinatising!

Definition. A *median* of a triangle is the line joining a vertex to the midpoint of the opposite edge.

Proposition 4. The three medians of a triangle intersect at the same point (called the *centroid* of the triangle).

Proof. Suppose the triangle is in $\mathbb{R}^2$. After applying a translation, we can assume that one of the vertices is at $(0, 0)$. If we then rotate about the origin, we can assume another one of the vertices is at $(2c, 0)$. After these transformations, suppose the third vertex is at $(2a, 2b)$. Then the three midpoints of the edges are (a, b), $(c, 0)$, and $(a + c, b)$. If we can prove that the three medians of this triangle intersect at the same point, we will have proved it for *every* triangle (can you see why?).

Let m_1 be the median passing through $(0, 0)$ and $(a + c, b)$. Then

$$m_1 = \{t(a + c, b) : t \in \mathbb{R}\}.$$

Similarly, if m_2 and m_3 are the medians passing through $(2a, 2b)$ and $(2c, 0)$ respectively we have

$$m_2 = \{(c, 0) + s(2a - c, 2b) : s \in \mathbb{R}\} \qquad \text{and} \qquad m_3 = \{(a, b) + r(2c - a, -b) : r \in \mathbb{R}\}.$$

Setting $t = \frac{2}{3}$ and $s = r = \frac{1}{3}$ shows that the point $\frac{2}{3}(a + c, b)$ lies on all three medians, completing the proof. ■

Exercise. An *altitude* of a triangle is the unique line perpendicular to an edge that passes through the opposite vertex. Prove that the three altitudes of a triangle intersect at a point (called the *orthocenter* of the triangle).

Lecture 3 - 05/15

1.1. Constructability

Some of the propositions in Euclid's Elements are propositions that we would be familiar with in modern mathematics. Statements that are true or false. Some of the propositions on the other hand, fall firmly on the constructions side of the fence. The construction of an equilateral triangle (Proposition 1) is an example of a construction.

The postulates and early propositions allow us to construct things using only a straight edge and compass. The things that can be constructed in Euclid's geometry are those which are possible using *only* these two tools. Our straight edge cannot have markings on it to indicate distance.

Why might we want to do a construction this way? For example, constructing an equilateral triangle like this? It turns out that it can often be more accurate than using a tool like a protractor. Let's see another example of such a construction.

Proposition 5 (Proposition 9, Book I). To bisect an angle.

Proof. Here's how the construction goes. Consider angle $\angle BAC$. Place the compass tip at A and draw a circle of any radius, which intersects AB and AC at D and E respectively. While keeping the radius of the compass fixed, draw two more circles centred at D and E. These two circles intersect at two points (can you see why the circles must intersect?). Draw a line through either one of the intersection points and A. This line bisects the angle. ■

Exercise. Prove that the construction does indeed give the angle bisector.

Exercise. Given two points, construct the midpoint using only a straight edge and compass.

Now, try as you might (and you should try!), it is impossible to *trisect* an arbitrary angle (that is, draw two lines which split the angle into three equal angles). As far as we can tell, ancient Greek mathematicians were trying (and presumably failing) to trisect an arbitrary angle. It wasn't until 1837 (around 2000 years after Euclid) that Pierre Wantzel proved that it was impossible.

This leads to a natural question: What can we construct using only a straight edge and compass? Here's a cool little story about regular polygons.

We have seen that we can construct an equilateral triangle. It turns out that you can also construct a square, and a regular pentagon. In 1796, Gauss constructed the regular 17-gon, in what is one of the truly amazing results, drawing on deep facts from number theory. In fact we know (well, sort of) *exactly* which regular n-gons are constructible.

Theorem 6 (Gauss-Wantzel 1837). A regular n-gon can be constructed if and only if $n = 2^k p_1 p_2 \ldots p_t$ where k is some non-negative integer and each p_i is a Fermat prime.

This theorem is some seriously deep mathematics, and was stated without proof by Gauss and proved by Wantzel. It *almost* gives a complete characterisation of which n-gons are constructable. The only problem is that we don't know which primes are Fermat primes, or even how many there are!

Definition. A *Fermat prime* is a prime of the form $2^{2^m} + 1$ where $m \geq 0$ is an integer.

Starting from $m = 0$ and going up the first few numbers we get are

$$3, 5, 17, 257, 65537$$

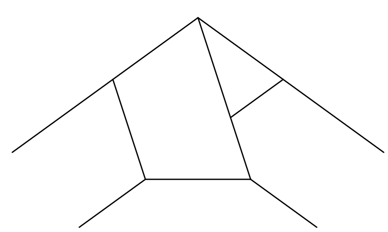
all of which are prime. However, when $m = 5$ we get 4294967297 which turns out not to be prime (it was shown by Euler in 1741 to be divisible by 641). In fact, we don't know of any other Fermat primes! It's currently an open question as to whether or not the five primes listed above are the only Fermat primes, and whether or not there are infinitely many Fermat primes. How very mysterious!

The proofs of non-constructability of various things (including, for example, a regular 9-gon) fall under the umbrella of Galois theory. Galois theory is an unreasonably beautiful, deep, and powerful branch of mathematics which you should certainly study if you get the chance!

1.2. Origami

There is a different way to construct things, using origami! Let's see some of this in action.

First, take a long thin rectangular strip of paper. After tying a simple knot it in, pulling it taut and creasing the knot, you can see a pentagon.



It turns out that this is a regular pentagon!

Exercise. Convince yourself, and your family, that the result of the above origami construction is indeed a regular pentagon.

It was mentioned earlier that you cannot trisect an arbitrary angle using a straight edge and compass. However, you can do it using origami! See the instructions at https://plus.maths.org/power-origami, under the section **Trisecting the angle**.

Exercise. Prove that the origami construction linked above does indeed trisect the angle. The trig identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ may come in handy here.

Lecture 4 - 05/20

2. Conic Sections

We now shift gears to the study of conic sections. Conic sections are the degree-two analogues of lines. There's a good chance you're already familiar with these. Examples of conic sections are hyperbolas, parabolas, ellipses (including circles), but also things like lines, points, and pairs of lines.

Roughly, we can think of these as being defined by degree-two polynomial equation. For example,

$$y = x^2, \quad \sqrt{2}x = y^2, \quad \text{and} \quad (x+y)^2 - x + y = -1$$

all define parabolas (plot them out, in desmos for example, if you don't believe me!). Two hyperbolas you may have seen are

$$x^2 - y^2 = 1, \quad xy = 1, \quad \text{and more generally} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for positive real numbers a and b. For some ellipses we have the general form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

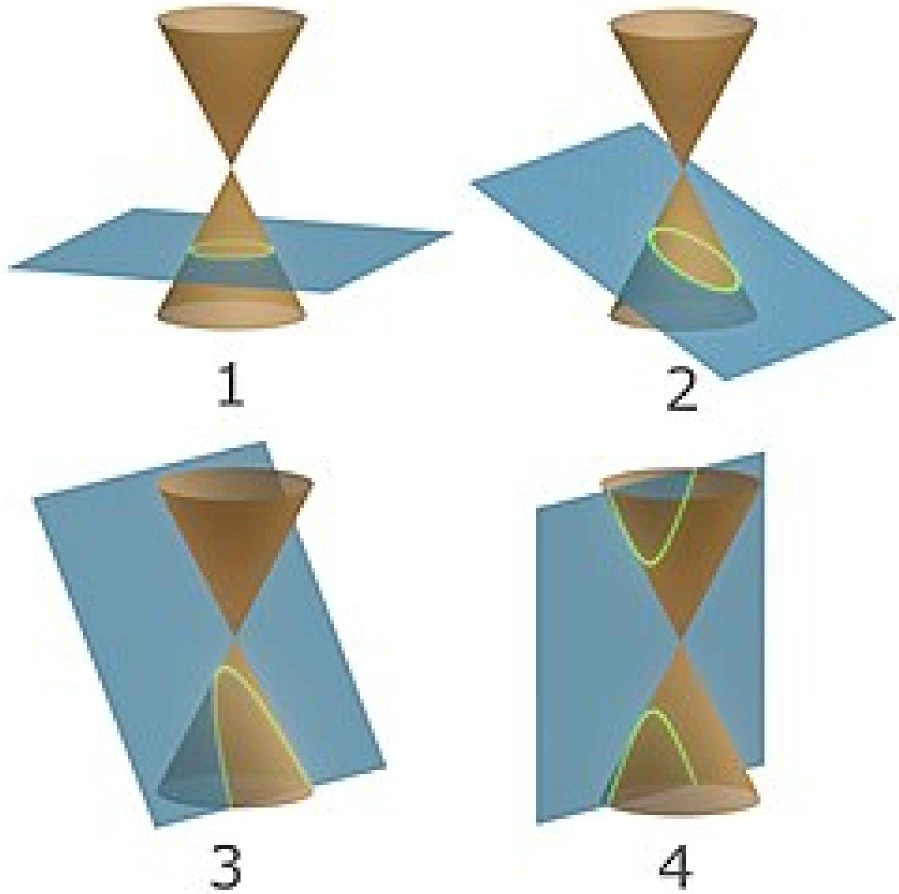
which is an ellipse centered at the origin, intersecting the x-axis at $\pm a$ and the y-axis at $\pm b$. When $a = b$ we get a circle centered at the origin with radius a. Ellipses, hyperbolas, and parabolas are what are known as *nondegenerate conics*. Examples of degenerate conics are points ($x^2 + y^2 = 0$ for example), lines ($(x+y)^2 = 0$ for example), and pairs of lines ($(x+y)(x-y) = 0$ or $(x+y)(x+y+1) = 0$ for example).

While the algebraic description as "something of degree two" is acceptable, it lacks the intrinsic geometric information we are after!

2.1. Conics as sections of cones.

Imagine an infinite icecream cone, balanced upon an upside-down infinite icecream cone. Then, with a sharp knife, make one cut through any two-dimensional plane in $\mathbb{R}^3$. The part where your knife cuts through the infinite cones will create a conic. Here is an image taken from Wikipedia[1].

[1]JensVyff, CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons



Depending on the angle of the cut, you get different conics. Image 1 shows a circle, Image 2 shows an ellipse, Image 3 a parabola, and Image 4 a hyperbola. Degenerate conics are obtained by cutting through the vertex of the cone, or by taking a slice tangent to the cone (in which case you get one line). The only conic mentioned above that you cannot get in this way is a pair of parallel lines, but we'll ignore that inconvenient little fact.

To make the connection between sections of cones and conics described algebraically, let's do some algebra. Consider the plane H defined by $z = 1$ in $\mathbb{R}^3$. Let's generate some cones that give different conics when intersected with H.

The circle $x^2 + y^2 = 1$ can be obtained by intersecting H with the cone in $\mathbb{R}^3$ defined by $x^2 + y^2 = z^2$. The parabola $y = x^2$ can be obtained by intersecting H with the cone in $\mathbb{R}^3$ defined by $yz = x^2$. The hyperbolas defined by $x^2 - y^2 = 1$ and $xy = 1$ are obtained by intersecting H with the cones defined by $x^2 - y^2 = z^2$ and $xy = z^2$ respectively. The degenerate conic that is a point is obtained by intersecting H with the cone $x^2 + y^2 = (z - 1)^2$.

If you're having trouble visualising these, I don't blame you! You can play around with a bunch of cones and their intersection with the plane H here on desmos.

Exercise. Consider the conic defined by the equation $3y^2 - 2xy - 4x^2 + 2x + 3y = -5$.

1. Graph this conic and identify its type (is it a parabola, ellipse, hyperbola, or something nondegenerate?).
2. Argue with your friends about what defines a cone in $\mathbb{R}^3$.
3. Find a cone in $\mathbb{R}^3$ whose intersection with the plane defined by $z = 1$ gives the conic.

2.2. Plane conics.

Now for a slightly different perspective, at least for ellipses, parabolas, and hyperbolas (which will turn out to be the same anyway!).

Definition. A *non-degenerate plane conic* is the set of points such that the distance to a fixed point f (called the *focus*) is equal to e times the distance to a fixed line D (called the *directrix*). The quantity $e > 0$ is called the *eccentricity*.

More symbolically, a non-degenerate plane conic is a subset of $\mathbb{R}^2$ of the form

$$\{x \in \mathbb{R} : \mathsf{dist}(x, f) = e\,\mathsf{dist}(x, D)\}.$$

We define the eccentricity of a circle to be 0. Recall that the distance between a point p and a line D is the distance between p and the point of intersection of D and the unique line orthogonal to D that passes through p.

Case 1: $e = 1$. Let $f = (a, 0)$ and D the line defined by $x = -a$, where $a > 0$. Let $p = (x, y)$ be a point on the conic. Then (x, y) satsifies

$$\begin{aligned} & \mathsf{dist}((a,0),(x,y)) = \mathsf{dist}((-a,y),(x,y)) \\ \iff \quad & (x-a)^2 + y^2 = (x+a)^2 \\ \iff \quad & x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2 \\ \iff \quad & y^2 = 4ax. \end{aligned}$$

So (x, y) is on the conic if and only if $y^2 = 4ax$, which is a parabola!

Exercise. Prove that all other conics with $e = 1$ can be translated and rotated to be of the form given above. One way to do this is to show that given *any* point and *any* line (not containing the point), there is a composition of a translation and a rotation that simultaneously sends the point and line to the point $(a, 0)$ and line defined by $x = -a$.

Lecture 5 - 05/22

Case 2: $0 < e < 1$. Let $f = (ae, 0)$, and let D be the line defined by $x = \frac{a}{e}$. For a point (x, y) on the conic we have

$$\begin{aligned} & \mathsf{dist}((ae,0),(x,y)) = e\,\mathsf{dist}\left(\left(\frac{a}{e}, y\right),(x,y)\right) \\ \iff \quad & (x-ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2 \\ & \vdots \ (\textit{algebraic shenanigans}) \\ \iff \quad & \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \end{aligned}$$

For simplicity, if we let $b = a\sqrt{1-e^2}$ so $b^2 = a^2(1-e^2)$ we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the equation of an ellipse! Note that since $0 < e < 1$, $0 < b < a$.

Exercise. Find another focus and directrix for the same ellipse.

Exercise. Let $e \in \mathbb{R}$ satisfy $0 < e < 1$, and let p and L be any point and line in the plane with $p \notin L$. Prove that there exists a positive real number a and a composition T of a translation and a rotation so that $T(p) = (ae, 0)$ and $T(L)$ is the line defined by $x = \frac{a}{e}$. Conclude that every conic with eccentricity between 0 and 1 differs from an ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by a composition of a translation and a rotation.

Case 3: $e > 1$. As in the ellipse case, let $f = (ae, 0)$ and let D be the line defined by $x = \frac{a}{e}$. Then we can again show (and you should make sure the algebra checks out) that (x, y) is on the conic if and only if

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b = a\sqrt{e^2 - 1}$. This is a hyperbola!

Exercise. Find another focus and directrix for the same hyperbola.

Exercise. Let $e \in \mathbb{R}$ satisfy $e > 1$, and let p and L be any point and line in the plane with $p \notin L$. Prove that there exists a positive real number a and a composition T of a translation and a rotation so that $T(p) = (ae, 0)$ and $T(L)$ is the line defined by $x = \frac{a}{e}$. Conclude that every conic with eccentricity $e > 1$ differs from a hyperbola defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

by a composition of a translation and a rotation.

Distance properties.

One way to draw a circle of radius r in real life, is to tie a string of length r between a pencil and a nail. Once the nail is fastened into some wood or a wall or something, you can place the pencil on the surface, and draw wherever it can go while the string is taut. The resulting shape will be a circle.

There is a similar way to draw an ellipse. You fasten a string between the two focal points of your desired ellipse, and make sure that the string is longer than the distance between the two focii (so that it has a bit of slack in it). Now push your pencil against the string, making the string taut. While keeping the string taut, move your pencil around. You will draw an ellipse!

Let's prove that this is what actually happens.

Theorem 7. For an ellipse E with foci f and f', $\operatorname{dist}(f,x)+\operatorname{dist}(f',x)$ is constant for all $x \in E$.

Proof. Let D and D' be the directrixes corresponding to f and f' respectively. Let c and c' be the points on D and D' satisfying $\operatorname{dist}(f,x) = e\operatorname{dist}(x,c)$ and $\operatorname{dist}(\text{f',x}) = e\operatorname{dist}(x,c')$. Since c', x, and c are collinear (can you see why?),

$$\operatorname{dist}(x,c)+\operatorname{dist}(x,c') = \operatorname{dist}(c,c') = \operatorname{dist}(D,D').$$

Therefore, $\operatorname{dist}(f,x)+\operatorname{dist}(f',x) = e\operatorname{dist}(D,D')$, which is constant and independent of $x \in E$. ■

Exercise. Figure out and prove the analagous theorem for hyperbolas.

Lecture 6 - 05/25

Reflection properties.

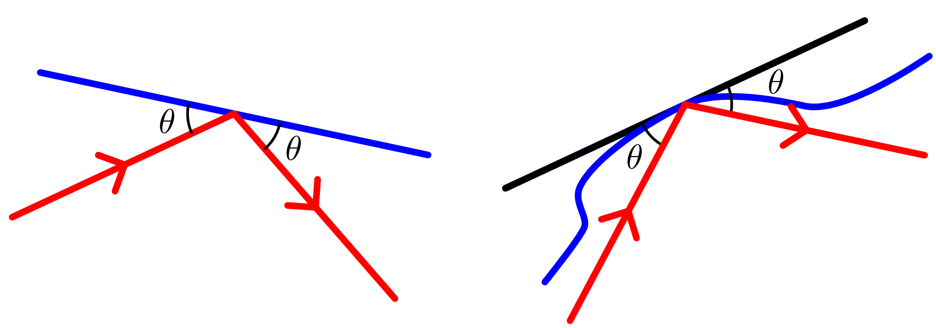
Conics are not merely mathematical curiosities. It turns out that they have incredibly useful reflection properties. As an example, a mirror in a telescope reflects incoming parallel light (from some impossibly far collection of stars) to a lens somewhere in the middle of space partially enclosed by the mirror. It turns out that in order to do this perfectly (or as close to perfectly as engineering allows), the mirror should be parabolic (and not spherical).

Another example of something physically cool like this (although arguably less useful) is the reflection property of an ellipse. If you are playing pool on an elliptical table with a hole at one of the focii, then if you hit a ball through the other focus, it is guaranteed to go in, either after hitting the wall once, or just directly! Of course, this is assuming that the ball bounces perfectly off the wall, and there are no physical effects like spin or friction screwing things up.

When light (or a particle under ideal physical conditions) reflects off a flat mirror, we have the following law that is obeyed:

Law of reflection: *The angle of incidence equals the angle of reflection.*

When the mirror is curved, the law of reflection still holds, except the angle of incidence and reflection are with respect to the tangent line at the point of reflection, as illustrated here:



So, in order to prove anything about reflections in conics, we need to get a handle on tangents. In order to do that, it will be very useful to parametrise our plane conics in terms of a single variable.

For a parabola E in standard form (recall $a > 0$ and $e = 1$) we have

$$E = \{(x,y) \in \mathbb{R}^2 : y^2 = 4ax\} = \{(at^2, 2at) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

For an ellipse E in standard form (here $a \geq b > 0$ and $b^2 = a^2(1-e^2)$) we have

$$E = \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} = \{(a\cos(t), b\sin(t)) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Finally, for a hyperbola E in standard form ($b^2 = a^2(e^2 - 1)$),

$$E = \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\} = \left\{ \left(\frac{a}{\cos(t)}, \frac{b\sin(t)}{\cos(t)} \right) \in \mathbb{R}^2 : t \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \right\}.$$

In each of these cases, as t varies, the point represented by the value of t moves around the conic in a continuous fashion. It's worth plotting out the points as t varies in the cases above. Can you see why we had to exclude the set $\left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$ from the values t can take on in the case of the hyperbola?

Exercise. Prove that all the set equalities defining the parametrisations of the conics are indeed equalities of sets.

Now that we have parametrisations, we can conveniently compute tangents.

Definition. The *tangent vector* to a curve defined parametrically by $(x(t), y(t))$ at a point $(x(t_0), y(t_0))$ is given by $(x'(t_0), y'(t_0))$.

That is, you just differentiate each coordinate function of t, with respect to t, and evaluate that derivative at the point of interest. Of course, the coordinate functions must be differentiable at the point in which you're interested to be able to compute the tangent vector at all.

One way to think of the tangent vector is to imagine a particle moving along a curve (as t varies). The tangent vector at a point p on the curve is the direction in which the particle would continue if suddenly at p it was no longer constrained to stay on the curve.

Conveniently, it also turns out that the tangent vector gives you the direction vector for the line tangent to the curve at that point.

Exercise. Consider the parabola $y^2 = 4x$, parametrised by $(t^2, 2t)$. Compute the tangent vector (v_1, v_2) at the point $(1, 2)$ on the parabola. By implicit differentiation, show that $\frac{dy}{dx}|_{(1,2)} = \frac{v_2}{v_1}$. This tells you that the tangent vector actually does point in the direction tangent to the parabola at $(1, 2)$.

Example. Consider the unit circle parametrised by $(\cos(t), \sin(t))$. The tangent vector at a point $(\cos(t), \sin(t))$ is $(-\sin(t), \cos(t))$. Notice that the dot product of the vector from the origin to a point on the circle with the tangent vector at that point is 0. This tells us the tangent line to the circle at any point is orthogonal to the line passing through that point and the center of the circle. A fact you likely have come across before! The corresponding fact about reflections here is that a billiards ball shot in any direction from the center of the circle will bounce off the wall and return to the center.

Lecture 7 - 05/27

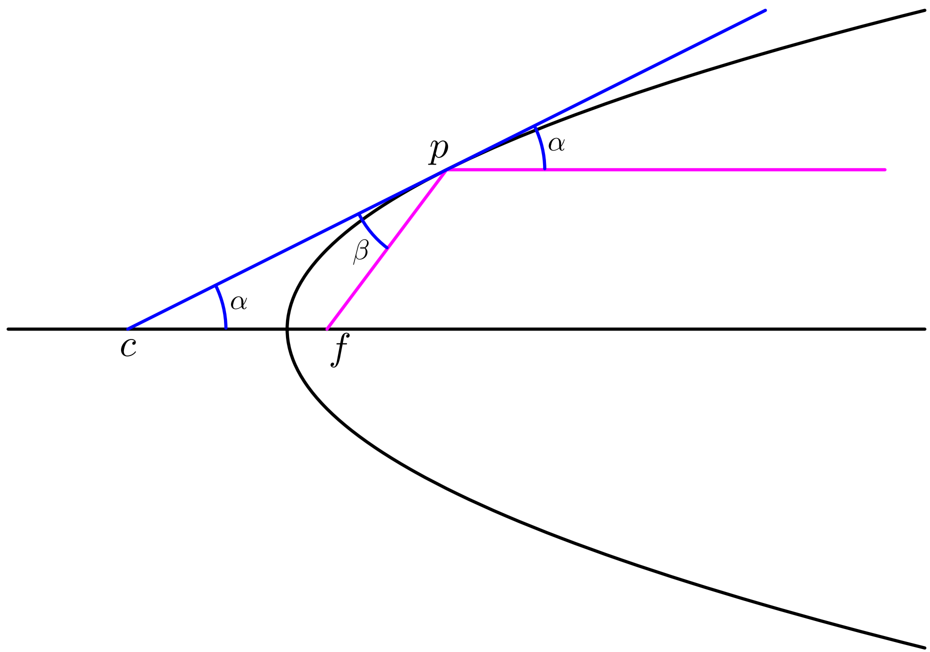
Proposition 8. Any line orthogonal to the directrix of a parabola reflects off the concave side of the parabola to the focus.

Proof. Consider a standard parabola defined by $y^2 = 4ax$, parametrised by $(at^2, 2at)$ and with focuse $f = (a, 0)$. Let $p = (at^2, 2at)$.

First note that if $t = 0$, the tangent line is vertical and a horizontal line on the x-axis will reflect back along the x-axis and pass through the focus. Since it's true for $t = 0$, let's assume $t \neq 0$.

The tangent vector at p is $(2at, 2a)$, so the slope of the tangent line at p is $\frac{1}{t}$. Since the tangent line passes through p, after a bit of algebra we see that the tangent line is defined by the equation $y = \frac{1}{t}x + at$. Let c be the x-intercept of the tangent line, so $c = (-at^2, 0)$.

We want to show that a horizontal ray of light will reflect off the concave side of the parabola at p and hit the focus f. To do this, we need to show that the angle α in the diagram below is equal to β.



To do this, it suffices to show that $\operatorname{dist}(c, f) = \operatorname{dist}(f, p)$, as then the triangle with vertices f, c, and p is isosceles.

We have

$$\begin{aligned}\operatorname{dist}(p, f) &= \sqrt{(at^2 - a)^2 + (2at)^2}\\ &= \sqrt{a^2 + 2a^2t^2 + a^2t^4}\\ &= a(1 + t^2)\\ &= a + at^2\\ &= \operatorname{dist}(c, f)\end{aligned}$$

completing the proof. ■

Similarly, there is a reflection property for the ellipse. Imagine playing billiards on a perfectly elliptical billiards table with a single hole at one of the foci. Then if you place a ball at the other focus point and shoot it in *any* direction, it will bounce off the wall of the elliptical table and go in the hole!

Proposition 9. Any line passing through one focus of an ellipse reflects off the ellipse and passes through the other focus.

The proof of Proposition 9 relies on the sine rule from triangles, which I'll remind you of first. For a triangle with side lengths A, B, and C, which subtend angles α, β, and γ respectively, we have

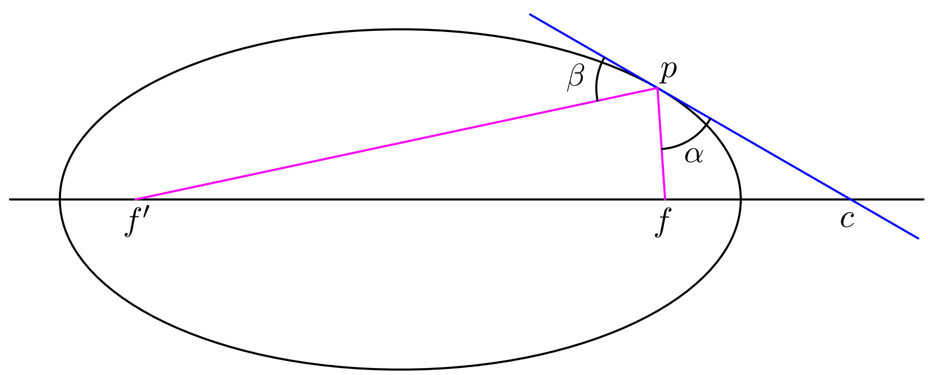
$$\frac{\sin(\alpha)}{A} = \frac{\sin(\beta)}{B} = \frac{\sin(\gamma)}{C}$$

Proof (of Proposition 9). Consider the standard ellipse defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b > 0$, and let p be a point on the ellipse given by $(a\cos(t), b\sin(t))$. If t is any multiple of $\frac{\pi}{2}$, the tangent line is horizontal or vertical, and it's an exercise for you to check the result holds in those cases. To simplify matters for us, we will assume that p is in the first quadrant, so $t \in \left(0, \frac{\pi}{2}\right)$. The other cases are similar and you can check the details if you wish.

The tangent vector at p is $(-a\sin(t), b\cos(t))$. The slope of the tangent line at p is therefore $\frac{-b\cos(t)}{a\sin(t)}$. Since the line passes through p we can deduce that the equation of the tangent line is

$$\frac{x}{a}\cos(t) + \frac{y}{b}\sin(t) = 1$$

which intersects the x-axis at $c = \left(\frac{a}{\cos(t)}, 0\right)$.



Let $f = (ae, 0)$ and $f' = (-ae, 0)$ be the focii with directixes defined by $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively. Then by the definition of an ellipse we have

$$\begin{aligned}
\operatorname{dist}(p, f) &= e\left(\frac{a}{e} - a\cos(t)\right) = a - ae\cos(t) \\
\operatorname{dist}(p, f') &= e\left(\frac{a}{e} + a\cos(t)\right) = a + ae\cos(t) \\
\operatorname{dist}(c, f) &= \frac{a}{\cos(t)} - ae = \frac{1}{\cos(t)}(a - ae\cos(t)) \\
\operatorname{dist}(c, f') &= \frac{a}{\cos(t)} + ae = \frac{1}{\cos(t)}(a + ae\cos(t))
\end{aligned}$$

Putting these observations together we have

$$\frac{\mathrm{dist}(p,f)}{\mathrm{dist}(c,f)} = \cos(t) = \frac{\mathrm{dist}(p,f')}{\mathrm{dist}(c,f')}.$$

Let θ be the angle between the tangent line and the x-axis. Then applying the sine rule to the triangles with vertices f, p, c and f', p, c gives

$$\frac{\sin(\theta)}{\sin(\alpha)} = \frac{\sin(\theta)}{\sin(\pi - \beta)} = \frac{\sin(\theta)}{\sin(\beta)}$$

and we can conclude $\alpha = \beta$, which is what we set out to prove. ∎

Click here for a fun numberphile video about playing billiards on an elliptical table

Exercise. Find and prove the analogous result for hyperbolas.

2.3. Plane conics vs conic sections

The relation between plane conics and conic sections is via something named Dandelin spheres. We will describe the situation in the case of an ellipse.

Consider a cone C and a plane H slicing through the cone to create an ellipse. There are two spheres that are simultaneously tangent to C and H. Consider one of these spheres S. The points that are on both S and C form a circle. There is a unique plane through this circle, call it L. Let f be the point at which S touches H, and let d be the line of intersection between L and H. Then f and d are one of the focus-directrix pairs for the ellipse! The other focus and directrix is obtained by using the other sphere.

There are similar constructions for the hyperbola and parabola, can you figure them out?

Lecture 8 - 05/29

3. Isometries of $\mathbb{R}^n$

We have used some transformations of the plane several times so far to make our lives easier. When we proved Proposition 2 about isosceles triangles, we made use of a reflection about a cleverly chosen line. When we dealt with computations surrounding conics, we first performed some transformation of $\mathbb{R}^2$ to put the conic in standard form.

In both of these cases, the reason we could do this is because the transformations we used preserved the information in which we were interested (namely distance and angle).

Exercise. Show that reflections across lines preserve distances and angles.

Definition. The *distance* between v and w in $\mathbb{R}^n$ is

$$\mathrm{dist}(v,w) = \|v - w\| = \sqrt{(v-w)\cdot(v-w)} = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + \cdots + (v_n - w_n)^2}$$

if $v = (v_1, v_2, ..., v_n)$ and $w = (w_1, w_2, ..., w_n)$.

Definition. An *isometry* of $\mathbb{R}^n$ is a function $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\mathrm{dist}(T(v), T(w)) = \mathrm{dist}(v, w)$ for all $v, w \in \mathbb{R}^n$.

Example. A reflection across a line in $\mathbb{R}^2$ is an isometry (this is an exercise).

Example. A *translation* $T(v) = v + a$ for some $a \in \mathbb{R}^n$. Let's check this is indeed an isometry. We have

$$\operatorname{dist}(T(v), T(w)) = \|T(v) - T(w)\| = \|v + a - w - a\| = \|v - w\| = \operatorname{dist}(v, w).$$

Orthogonal transformations.
An important family of isometries are orthogonal transformations. Recall that an $n \times n$ real matrix is *orthogonal* if any of the following conditions hold:

- $Av \cdot Aw = v \cdot w$ for all $v, w \in \mathbb{R}^n$.
- $A^T = A^{-1}$.
- The columns of A form an orthonormal basis of $\mathbb{R}^n$ with respect to the dot product.
- The rows of A form an orthonormal basis of $\mathbb{R}^n$ with respect to the dot product.

An *orthogonal transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ is one that comes from multiplication by an orthogonal matrix. That is, $T(v) = Av$ for some orthogonal matrix A. Orthogonal transformations are isometries since

$$\begin{aligned}
\operatorname{dist}(Av, Aw) &= \sqrt{(Av - Aw) \cdot (Av - Aw)} \\
&= \sqrt{Av \cdot Av - 2Av \cdot Aw + Aw \cdot Aw} \\
&= \sqrt{v \cdot v - 2v \cdot w + w \cdot w} \\
&= \sqrt{(v - w)v \dot{-} w} \\
&= \operatorname{dist}(v, w).
\end{aligned}$$

In the 2×2 case, the only possibilities for orthogonal matrices are

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}.$$

The former is a rotation by θ counterclockwise about the origin, and the latter is the reflection across the line which makes an angle of $\frac{\theta}{2}$ with the positive x-axis (can you see why?). These are the only possibilities!

Lecture 9 - 06/01

Example. Perhaps the most important example, and maybe the least exciting, of an isometry of $\mathbb{R}^n$ is the identity map $\operatorname{Id} : \mathbb{R}^n \to \mathbb{R}^n$, which is defined by $\operatorname{Id}(v) = v$ for all $v \in \mathbb{R}^n$. You should check that it is indeed an isometry (even if you think it's obvious, because sometimes obvious things are false!).

Here's a surprisingly powerful application of thinking about an angle θ not as a static measurement, but as a measure of how much you rotate something to land on the other thing. Since rotation about the origin is a linear map, we can represent an angle as a rotation matrix by that angle.

So far this probably doesn't make much sense, so let me be a little more precise. Let R_θ be the rotation in $\mathbb{R}^2$ about the origin by an angle of θ counterclockwise. Then composing such rotations results in a rotation by the sum of the angles. Nothing terribly controversial here. Symbolically, $R_\theta \circ R_\varphi = R_{\theta+\varphi}$.

Now, since each rotation about the origin is represented by a matrix, composition of rotations is obtained by matrix multiplication. After all, matrix multiplication is defined to make multiplication play nicely with composition of functions. We have

$$\begin{aligned}\begin{bmatrix}\cos(\theta+\varphi) & -\sin(\theta+\varphi)\\ \sin(\theta+\varphi) & \cos(\theta+\varphi)\end{bmatrix} &= \begin{bmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\\ &= \begin{bmatrix}\cos(\varphi)\cos(\theta)-\sin(\varphi)\sin(\theta) & -\cos(\varphi)\sin(\theta)-\sin(\varphi)\cos(\theta)\\ \cos(\varphi)\sin(\theta)+\sin(\varphi)\cos(\theta) & \cos(\varphi)\cos(\theta)-\sin(\varphi)\sin(\theta)\end{bmatrix}.\end{aligned}$$

Perhaps surprisingly, the angle sum formulas are now staring us in the face! For example, the top left entry of the first and last matrices in the previous string of equalities tells us

$$\cos(\theta+\varphi) = \cos(\varphi)\cos(\theta) - \sin(\varphi)\sin(\theta).$$

Cool. You should see what other trig identities you can recover from this point of view.

Speaking of angles, the study of isometries was motivated by saying that isometries don't change the things in which we're interested, the things being length and angles. The definition of an isometry only insists that the function preserves length. It turns out that if you preserve lengths, you also preserve angles.

Proposition 10. Angles between lines are preserved by isometries. More precisely, let $x, y, z \in \mathbb{R}^2$ with $\angle yxz = \alpha$ and let T be an isometry of $\mathbb{R}^2$. Then $\angle T(y)T(x)T(z) = \alpha$.

Proof. This proposition follows from the cosine law. Recall that for a triangle with side lengths a, b, c, and angle θ opposite the side with side length a, then

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc}.$$

What the cosine law tells us is that angles are entirely determined by lengths. It is now an exercise for you to fill in the rest of the details of this proof! ■

Great, so isometries preserve lengths *and* angles. Now, we have seen in various examples that if you take two isometries and compose them, you end up with an isometry. Perhaps this is immediately believable, but as always, the details should be checked.

Proposition 11. The composition of two isometries of $\mathbb{R}^n$ is an isometry of $\mathbb{R}^n$.

Proof. Let T and S be isometries of $\mathbb{R}^n$ and let $v, w \in \mathbb{R}^n$. Then

$$\text{dist}(T(S(v)), T(S(w))) = \text{dist}(S(v)), S(w)) = \text{dist}(v, w)$$

where the first equality is since T is an isometry, and the second is because S is an isometry. ■

Good! It's always nice when our intuition agrees with the mathematics. Here are some exercises to get warmed up with composing isometries.

Exercise. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation counterclockwise by θ around a point $p \in \mathbb{R}^2$. Show that there exist lines l_1 and l_2 in $\mathbb{R}^2$ so that $R = \sigma_{l_2} \circ \sigma_{l_1}$, where σ_l is the reflection across l.

Example. Let l_1 and l_2 be two parallel lines a distance d apart. For the sake of this example, let's draw l_1 and l_2 as vertical lines, with l_1 to the left of l_2. What is the composition $T = \sigma_{l_2} \circ \sigma_{l_1}$?

Let's prod it by looking at what happens to some points. Let's first take x on l_1. Then $\sigma_{l_1}(x) = x$, and σ_{l_2} moves x to the right by a distance of $2d$.

If x is on l_2, then $\sigma(l_1)$ first moves x to the left by $2d$, and then σ_{l_2} moves $\sigma_{l_1}(x)$ to the right by $4d$. So, $T(x)$ is the point a distance $2d$ to the right of x.

Now let's suppose x is a point distance a to the left of l_1, where $a < d$. Then $\sigma_{l_1}(x)$ is a to the right of l_1, and $(d-a)$ to the left of l_2. Applying the reflection across l_2 then moves $\sigma_{l_1}(x)$ to the point $(d-a)$ to the right of l_2. So, $T(x)$ moves x a distance of $a+a+(d-a)+(d-a) = 2d$ to the right.

Although we have not covered all possible starting points for x, perhaps you are starting to be convinced that T is the translation to the right by a distance of $2d$. The important thing to note here is that the translation distance does not depend on a!

Exercise. Show that the composition of two reflections across parallel lines is a translation.

Exercise. Let m be a line in $\mathbb{R}^2$ orthogonal to two distinct parallel lines l_1 and l_2. Is the isometry $\sigma_m \circ \sigma_{l_1} \circ \sigma_{l_2}$ a rotation, reflection, translation, or something else?

Lecture 2 - 06/03

Great, we have some fundamental facts about isometries under our belts, let's shift our focus to understanding what general isometries look like.

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation about the point $(0,1)$ by $\frac{\pi}{2}$ counterclockwise. How can we write down a formula for T? One useful strategy for this is to compose a few isometries together. Let S_1 be the translation of $\mathbb{R}^2$ that maps $(0,1)$ to $(0,0)$, R the rotation about $(0,0)$ by $\frac{\pi}{2}$ counterclockwise, and S_2 the translation taking $(0,0)$ back to $(0,1)$. Then it should be case that $T = S_2 \circ R \circ S_1$. Let's use this to find a formula for T. For $v \in \mathbb{R}^2$ we have

$$\begin{aligned} T(v) &= S_2 \circ R \circ S_1(v) \\ &= S_2 \circ R\left(v - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= S_2\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(v - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(v - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v + \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

you should check that this indeed does give the desired isometry. Let's check that there's at least a chance. A rotation about $(0,1)$ should at the very least, fix $(0,1)$. We have

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which is great.

Exercise. Prove that the formula given in the previous example does actually describe a rotation about $(0,1)$ by $\frac{\pi}{2}$ counterclockwise. Part of this exercise is figuring out what this means.

Curiously, the formula suggest another way to think about the rotation about $(0,1)$. You first rotate by $\frac{\pi}{2}$ counterclockwise about the origin, and then translate by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Weird. Draw some pictures to convince yourself this isn't totally ludicrous.

The previous example suggests a form in which we could write every isometry. We will prove that this is indeed the case in a few a steps.

Lemma 12. An isometry T of $\mathbb{R}^n$ can be uniquely written as $T(v) = S(v) + w$ where $w \in \mathbb{R}^n$ and S is an isometry of $\mathbb{R}^n$ fixing the origin 0 (that is, $S(0) = 0$).

Proof. Let $w = T(0)$ and define $S(v) = T(v) - T(0)$. Then $w \in \mathbb{R}^n$, $S(v)$ is an isometry (since it is a composition of T and a translation) and $S(0) = T(0) - T(0) = 0$. For uniqueness, suppose $T(v) = S'(v) + w'$ where S' is an isometry of $\mathbb{R}^n$ fixing 0 and $w' \in \mathbb{R}^n$. Then

$$\begin{aligned} & S'(0) + w' = S(0) + w \\ \Rightarrow \quad & 0 + w' = 0 + w \\ \Rightarrow \quad & w' = w. \end{aligned}$$

Then, for all $v \in \mathbb{R}^n$,

$$\begin{aligned} & S'(v) + w' = S(v) + w \\ \Rightarrow \quad & S'(v) = S(v) \end{aligned}$$

and so S' and S are equal isometries. ∎

The next piece we need is something that we will take as a fact without proof.

Fact 13. If T is an isometry of $\mathbb{R}^n$ that fixes the origin, then $T(v) = Av$ for all $v \in \mathbb{R}^n$, where A is some orthogonal matrix.

Putting together Lemma 12 and Fact 13 we get the following theorem.

Theorem 14. For every isometry T of $\mathbb{R}^n$, there exists a unique $n \times n$ orthogonal matrix and a unique $w \in \mathbb{R}^n$ so that $T(v) = Av + w$ for all $v \in \mathbb{R}^n$.

Quick diversion: Groups

Let $\mathrm{Isom}(\mathbb{R}^n)$ be the set of all isometries of $\mathbb{R}^n$. So far we have seen that there is a way to eat two isometries in $\mathrm{Isom}(\mathbb{R}^n)$ and get another isometry (composition). Even better, there is a special isometry $\mathrm{Id} \in \mathrm{Isom}(\mathbb{R}^n)$ that satisfies $T \circ \mathrm{Id} = \mathrm{Id} \circ T = T$ for all $T \in \mathrm{Isom}(\mathbb{R}^n)$.

There is a perspective which allows us to study $\mathrm{Isom}(\mathbb{R}^n)$ as a set of things with a operation. The set being the set of isometries, and the operation being composition (much like the integers have addition). That perspective is the perspective of groups.

Definition. A *group* is a set G with an operation $\circ$ satisfying

1. There is an *identity element* $e \in G$ so that $e \circ g = g \circ e = g$ for all $g \in G$,
2. For all $g, h, k \in G$, $(g \circ h) \circ k = g \circ (h \circ k)$.
3. For every $g \in G$, there is an element $g^{-1} \in G$ (called the *inverse* of G) satisfying $g \circ g^{-1} = g^{-1} \circ g = e$.

For us, the set G is $\mathrm{Isom}(\mathbb{R}^n)$ and the operation $\circ$ is composition of isometries. Property 1 holds as was mentioned above. Property 2 holds because composition of functions always behaves that way. In order to show that $\mathrm{Isom}(\mathbb{R}^n)$ with composition is a group, we need to show that every isometry has an inverse. Intuitively, for any isometry we're looking for one that undoes what was just done.

Let $T \in \mathrm{Isom}(\mathbb{R}^n)$ and write $T(v) = Av + w$ for an $n \times n$ orthogonal matrix A. Let

$$S(v) = A^{-1}v - A^{-1}w.$$

Then for all $v \in \mathbb{R}^n$,

$$T \circ S(v) = T\big(A^{-1}v - A^{-1}w\big) = A\big(A^{-1}v - A^{-1}w\big) + w = v$$

and similarly (with details left to you to check)

$$S \circ T(v) = v.$$

Therefore $S \circ T(v) = T \circ S(v) = \mathrm{Id}(v)$ for all $v \in \mathbb{R}^n$, which is precisely what it means for the isometries $S \circ T$, $T \circ S$, and Id to be equal. We can conclude that $S = T^{-1}$ and consequently, $\mathrm{Isom}(\mathbb{R}^n)$ is a group!

The main takeway from this is that elements of groups have inverses. Said in our context, every isometry has an isometry that reverses it.

Example. Let σ_l be a reflection across a line l in $\mathbb{R}^2$. Then since $\sigma_l \circ \sigma_l(v) = v$ for all $v \in \mathbb{R}^2$, $\sigma_l \circ \sigma_l = \mathrm{Id}$ and $\sigma_l^{-1} = \sigma_l$. Reflections are their own inverses!

Let $S, T \in \mathrm{Isom}(\mathbb{R}^n)$. Suppose we want to find the inverse of $S \circ T$. Intuitively, this isometry is achieved by performing T followed by S. So if we undo S and then undo T (in that order), we should undo the composition. More formally, for all $v \in \mathbb{R}^n$ we have

$$(S \circ T) \circ (T^{-1} \circ S^{-1})(v) = S\big(T\big(T^{-1}\big(S^{-1}(v)\big)\big)\big) = S\big(S^{-1}(v)\big) = v$$

and

$$(T^{-1} \circ S^{-1}) \circ (S \circ T)(v) = T^{-1}\big(S^{-1}\big(S(T(v))\big)\big) = T^{-1}(T(v)) = v.$$

Therefore, $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. This should look vaguely familiar to a property relating matrix multiplication and inverses.

Lecture 11 - 06/05

At this point, we've learnt quite a lot about what isometries can look like. But, like all things in mathematics, there's more to learn. We have so far seen that rotations and translations can both be written as compositions of reflections. It turns out that *every* isometry is a composition of reflections. Let's focus on this in the case of isometries of $\mathbb{R}^2$.

3.1. Isometries of $\mathbb{R}^2$

One approach to understanding isometries of $\mathbb{R}^2$ is to understand how isometries deal with triangles.

Definition. Two triangles in $\mathbb{R}^2$ with vertices A, B, C and A', B', C' are *congruent* if

$$\begin{aligned}\operatorname{dist}(A,B) &= \operatorname{dist}(A',B')\\ \operatorname{dist}(A,C) &= \operatorname{dist}(A',C'), \quad \text{and}\\ \operatorname{dist}(B,C) &= \operatorname{dist}(B',C').\end{aligned}$$

Understanding isometries as compositions of reflections relies on the following theorem.

Theorem 15. Given two congruent triangles with vertices A, B, C and A', B', C' respectively, there is a unique $T \in \operatorname{Isom}(\mathbb{R}^2)$ satisfying $T(A) = A', T(B) = B'$, and $T(C) = C'$.

Proof (Sketch). Here is a sketch of the proof of the existence of such an isometry. The details and existence proof are left to you.

Let T_1 be the translation so that $T_1(A) = A'$. Let T_2 be the reflection across the line m which is the perpendicular bisector of $T(B)$ and B'. Since $T(A) = A'$ and $\operatorname{dist}(T(A), T(B)) = \operatorname{dist}(A', B')$, it can be shown that $A' = T(A)$ lies on m. Then $T_2 \circ T_1(A) = A'$ and $T_2 \circ T_1(B) = B'$.

If $T_2 \circ T_1(C) = C'$, we're done. If not, let T_3 be the reflection across the line containing A' and B'. Then $T = T_3 \circ T_2 \circ T_1$ is the isometry we're after.

Uniqueness is left as an exercise. ■

Exercise. Turn the above proof sketch into a proof (in particular, you will need to check that all the claims made in the sketch are indeed true).

Theorem 16. Every isometry of $\mathbb{R}^2$ is a composition of at-most three reflections.

Proof. Let $T \in \operatorname{Isom}(\mathbb{R}^2)$ and choose any three non-collinear points A, B, and C. Then the triangle with vertices A, B, and C is congruent to the triangle with vertices $A' = T(A)$, $B' = T(B)$, and $C' = T(C)$. Note that T must be the unique isometry satisfying $T(A) = A'$, $T(B) = B'$, and $T(C) = C'$.

Case 1: If $A = A'$, $B = B'$ and $C = C'$, then the identity isometry Id satisfies $\operatorname{Id}(A) = A', \operatorname{Id}(B) = B'$, and $\operatorname{Id}(C) = C'$. Therefore, $T = \operatorname{Id}$ and is a composition of reflections (namely zero reflections).

Case 2: Suppose $A = A'$, $B = B'$, and $C \neq C'$. Then C and C' are the two distinct points of intersection between the circles with centers A and B and radii $\operatorname{dist}(A, C)$ and $\operatorname{dist}(B, C)$ respectively. Let l be the line passing through A and B. Then $\sigma_{l(A)} = A'$, $\sigma_{l(B)} = B'$ and $\sigma_{l(C)} = C'$. Therefore, $T = \sigma_l$ and is a composition of reflections (namely one reflection).

Case 3: Suppose $A = A'$, $B \neq B'$, and $C \neq C'$. Let m be the perpendicular bisector of B and B'. Then m passes through A (can you see why?) and so $\sigma_{m(A)} = A'$ and $\sigma_{m(B)} = B'$, which gets us back into either Case 1 or Case 2. In Case 1 or Case 2, we need at most one reflection. Composing that with σ_l shows that in this case, T is a composition of at most two reflections.

Case 4: Assume $A \neq A'$, $B \neq B'$, and $C \neq C'$. Let σ be a reflection so that $\sigma(A) = A'$. Now we're in one of the previous cases. In all the previous cases, at most two reflections were needed. So in this case T is a composition of at most three reflections. ■

Exercise. Prove that every isometry of $\mathbb{R}$ is a composition of at most two reflections.

Satisfyingly, this result generalises to $\mathbb{R}^n$. Here is the result which we will not prove in this course (but it's interesting to think how you might do so!). A reflection in $\mathbb{R}^n$ is a reflection across an $(n-1)$-dimensional plane.

Fact 17. Every isometry of $\mathbb{R}^n$ is a composition of at most $n+1$ reflections. If an isometry fixes a point, it is the composition of at most n reflections.

Lecture 12 - 06/08 - Quiz 1

Lecture 13 - 06/10

We now know that every element of $\mathrm{Isom}(\mathbb{R}^2)$ is a composition of at most three reflections. Here is the situation broken down even more.

Zero reflections
This is the identity isometry.

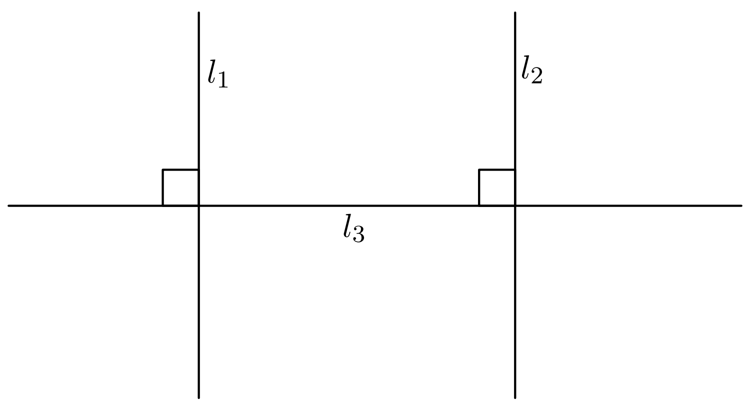
One reflection
This is, well, a reflection.

Two reflections
Here things get a tiny bit more interesting. We've seen earlier that if the two lines intersect, the composition of the two reflections is a rotation (with rotation center the point of intersection).

If the two lines are parallel, an exercise shows that the composition is a translation.

Three reflections
If an isometry cannot be written as a composition of 0, 1, or 2 reflections, then it is what is called a *glide reflection*. A glide reflection is a composition of a translation by some vector w, and a reflection across a line parallel to w.

To realise a glide reflection as a composition of three reflections, let l_1 and l_2 be two parallel lines, and let l_3 be orthogonal to both l_1 and l_2.



Then $\sigma_{l_2} \circ \sigma_{l_1}$ is a translation (as above) in a direction parallel to l_3. If we then compose with the reflection across l_3 we get the glide reflection $\sigma_{l_3} \circ \sigma_{l_2} \circ \sigma_{l_1}$.

To picture a glide reflection, imagine walking barefoot on the beach (and assume that your feet are perfect reflections of each other, like in the image below[2]). Then to map one of your left footprints to one of your right footprints, you can perform a glide reflection by translating in the direction you are walking followed by a reflection across the line that runs in the direction you are walking, and goes right between your two feet.

Exercise. Show that a glide reflection is not secretly a reflection, rotation, or translation. *Hint*: Look at what is fixed by each type of isometry.

Orientation-preserving and reversing isometries

With our newfound knowledge of isometries as compositions of reflections safely stowed in our backpacks, let's return to the point of view bestowed upon us by Theorem 14.

Suppose $S, T \in \mathrm{Isom}(\mathbb{R}^n)$ are of the form

$$S(v) = Av + w \quad \text{and} \quad T(v) = Bv + z$$

for $n \times n$ orthogonal matrices A and B, and $w, z \in \mathbb{R}^n$. Then

$$S \circ T(v) = A(Bv + z) + w = ABv + (Az + w).$$

The important thing to notice here is that the unique orthogonal matrix corresponding to $S \circ T$ is the product of the orthogonal matrices corresponding to S and T.

We can now lean on two extremely convenient facts from linear algebra (it's a good idea to remind yourself, or figure out, how to prove them):
1. $\det(AB) = \det(A)\det(B)$
2. If A is orthogonal, then $\det(A) = 1$ or -1.

We can now sort the isometries of $\mathbb{R}^n$ into two bins; those represented by an orthogonal matrix with determinant 1, and those represented by an orthogonal matrix with determinant -1.

Definition. Define the set of *orientation-preserving isometries of* $\mathbb{R}^n$ as

$$\mathrm{Isom}_+(\mathbb{R}^n) = \{T \in \mathrm{Isom}(\mathbb{R}^n) : T(v) = Av + w \text{ and } \det(A) = 1\}$$

An isometry is *orientation-reversing* if it is not an orientation-preserving isometry.

Exercise. Show that $\mathrm{Isom}_+(\mathbb{R}^n)$ with composition as the operation is a group.

Let's dig out our isometries-as-compositions-of-reflections perspective from our backpacks and return to isometries of $\mathbb{R}^2$.

Suppose l is a line passing through a point $p \in \mathbb{R}^2$ and forms an angle of θ with the positive x -axis. Then we can deduce the formula for the reflection σ_l by first translating p to the origin,

[2]By Д.Ильин: vectorization - File:Krok 6.png by Poznaniak, CC0, https://commons.wikimedia.org/w/index.php?curid=155808009

performing the reflection about a line through the origin parallel to our original line, and then translating the origin back to p. This gives

$$\sigma_l(v) = A(v-p) + p = Av + (p - Ap)$$

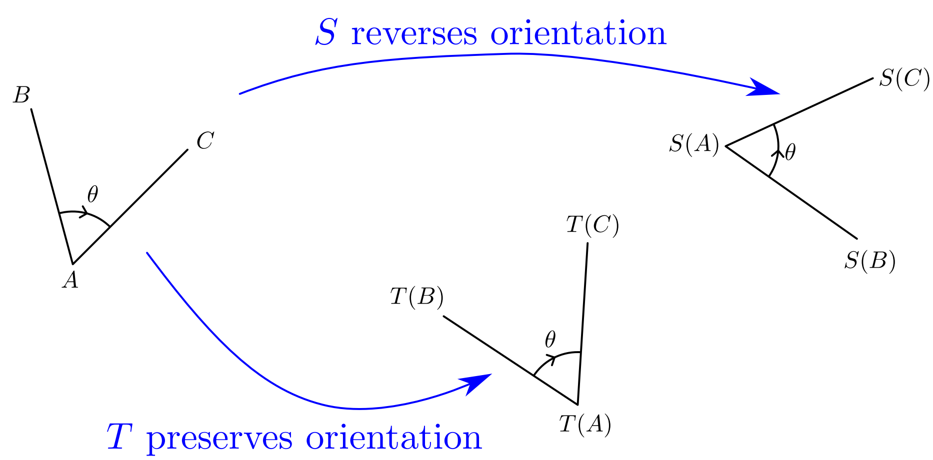
where $A = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ (recall that A is the matrix representing a reflection across the line through the origin that forms an angle of θ with the positive x-axis).

The main thing to notice here is that $\det(A) = -1$. So, a reflection is orientation-reversing. Rotations and translations are compositions of two reflections, so the corresponding orthogonal matrix will be a product of two matrices with determinant -1. Therefore, rotations and translations are orientation-preserving isometries. Similarly, glide-reflections are orientation reversing. Continuing further we can argue that the composition of an even number of reflections is an orientation-preserving isometry, and the composition of an odd number of reflections is an orientation-reversing isometry.

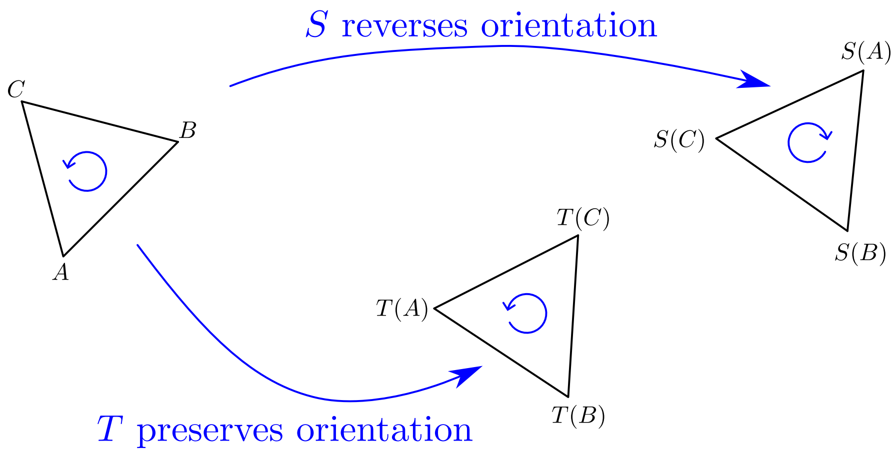
Exercise. You can write the identity as the composition of two reflections (choose any reflection σ and note that $\mathrm{Id} = \sigma \circ \sigma$. Can you write the identity as the composition of seven reflections?

There are a couple of ways to interpret the phrase "orientation preserving" in $\mathbb{R}^2$.

We know isometries preserve angles, but they don't necessarily preserve the direction in which the angle is measured (think about what a reflection does to angles). Orientation-preserving isometries not only preserve the measure of angles, but also the direction in which the angle is measured.



Another way to think about orientation is by keeping track of the order of the vertices of a triangle as you move counterclockwise around the triangle. More explicitly, suppose a triangle has vertices A, B, and C, and if you start at A and travel counterclockwise around the interior of the triangle, you first meet B and then C. If $T \in \mathrm{Isom}_+(\mathbb{R}^2)$, then the same will hold of the triangle with vertices $T(A)$, $T(B)$, and $T(C)$. That is, if you start at $T(A)$ and go counterclockwise around the triangle, you will first meet $T(B)$, and then $T(C)$. If starting at $T(A)$ and going counterclockwise you meet $T(C)$ before $T(B)$, then T reverses orientation.



Exercise. We know that every isometry of $\mathbb{R}^2$ is a composition of reflections. Suppose $T = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_k$ is an isometry written as a composition of reflections. Prove that $T \in \mathrm{Isom}_+(\mathbb{R}^n)$ if and only if k is even.

Lecture 14 - 06/12 - David McKinnon tiling lecture

Lecture 15 - 06/15

4. Interior Design

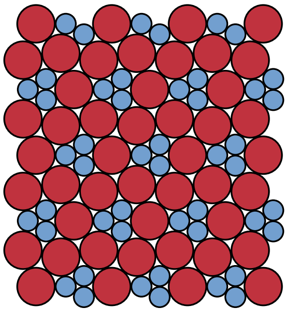
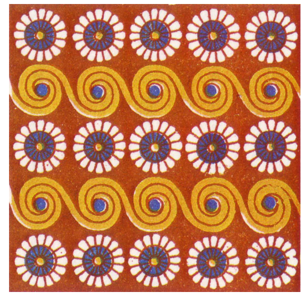
It turns out that interior (and exterior) design is an amazingly rich source of mathematics relating to isometries of $\mathbb{R}^2$.

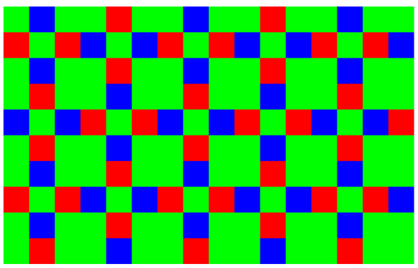
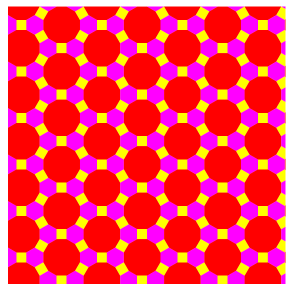
4.1. Wallpaper patterns

(The treatment of this section follows closely the wonderful book *The Magic Theorem* by John H. Conway, Heidi Burgiel, and Chaim Goodman-Strauss.)

When decorating a wall or a floor, it's very common to have a repeating pattern. Besides being somewhat aesthetically pleasing, it helps with cost and efficiency of construction to repeat elements over and over again. The question we are going to address is whether or not two wallpaper patterns have the same repeating pattern (whatever that means).

To get started, here are some examples of wallpaper patterns:

(and the image credits from left to right, top to bottom[3][4][5][6][7][8][9]).

Without being too precise, what we mean by a wallpaper pattern is something which has repeating patterns covering the entire plane. I won't be any more precise than this in these notes.

What exactly do we mean by different wallpaper patterns here. Are the wallpaper patterns above fundamentally different in some way, or are they really the same pattern just with different colours and shapes? Our main tool to tell apart wallpaper patterns will be to study their symmetries.

Definition. A *symmetry* of a wallpaper pattern is an isometry of $\mathbb{R}^2$ that leaves the wallpaper pattern unchanged. Given a wallpaper pattern, the set of all symmetries of that pattern is called a *wallpaper group*.

As the name suggests, a wallpaper group is indeed a group. So, in particular, the identity is a symmetry of every wallpaper pattern, if two symmetries of a wallpaper pattern are composed, you get another symmetry, and the inverse of every symmetry is again a symmetry.

We will investigate the possible wallpaper patterns (or what turns out to be the same thing, the possible wallpaper groups), by looking for certain features that are built out of collections of elements of the wallpaper groups.

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[3] By Toby Hudson - Own work, following Tom Kennedy [2], [3], CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=10503540

[4] By © Nevit Dilmen, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=28095443

[5] By Owen Jones, uploaded to the English Wikipedia by w:en:User:Dmharvey - The Grammar of Ornament. Egyptian No 7 (plate 10), image 20, Public Domain, https://commons.wikimedia.org/w/index.php?curid=3016559

[6] By Owen Jones - en:Image:Wallpaper_group-p1-3.jpg, Public Domain, https://commons.wikimedia.org/w/index.php?curid=2979174

[7] By Ckendel at German Wikipedia - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=19113240

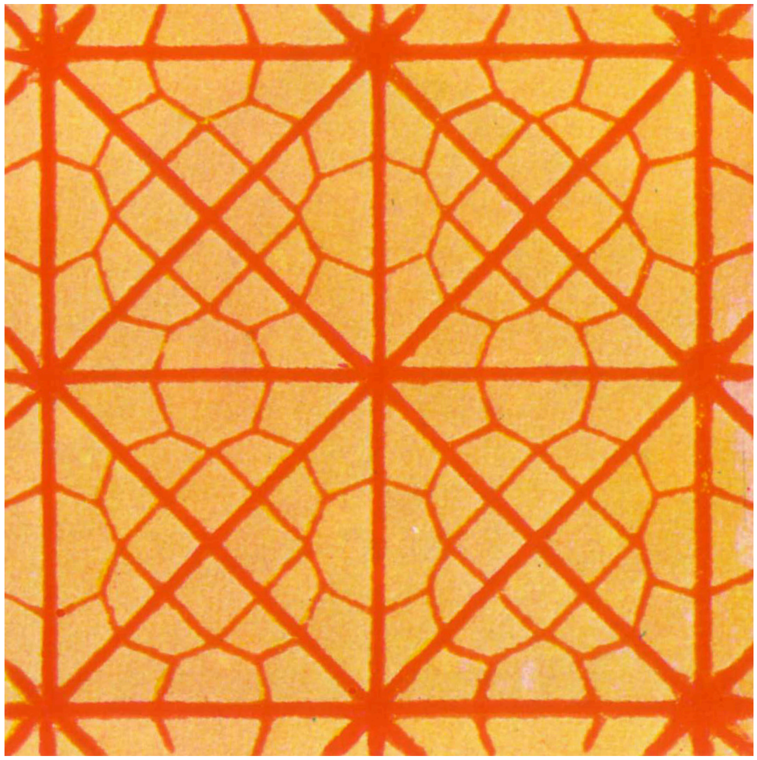
[8] none

[9] Public Domain, https://commons.wikimedia.org/w/index.php?curid=3022628

Lecture 16 - 06/17

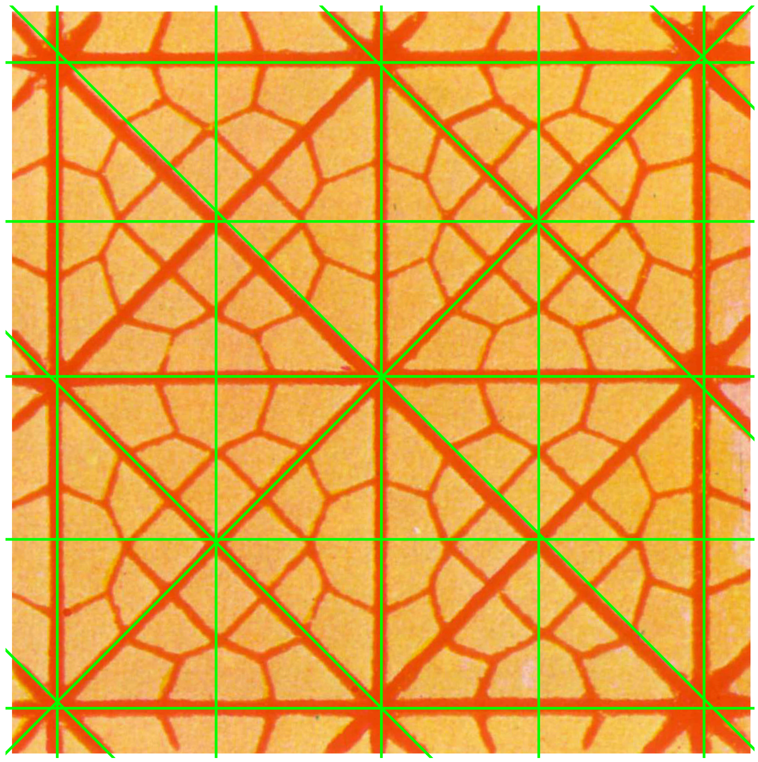
Feature 1 - Kaleidoscopes

The first thing to look for when probing a wallpaper pattern are Kaleidoscopes, or lines of reflection. Consider the following wallpaper pattern[10] (and assume the pattern repeats forever, covering the plane).

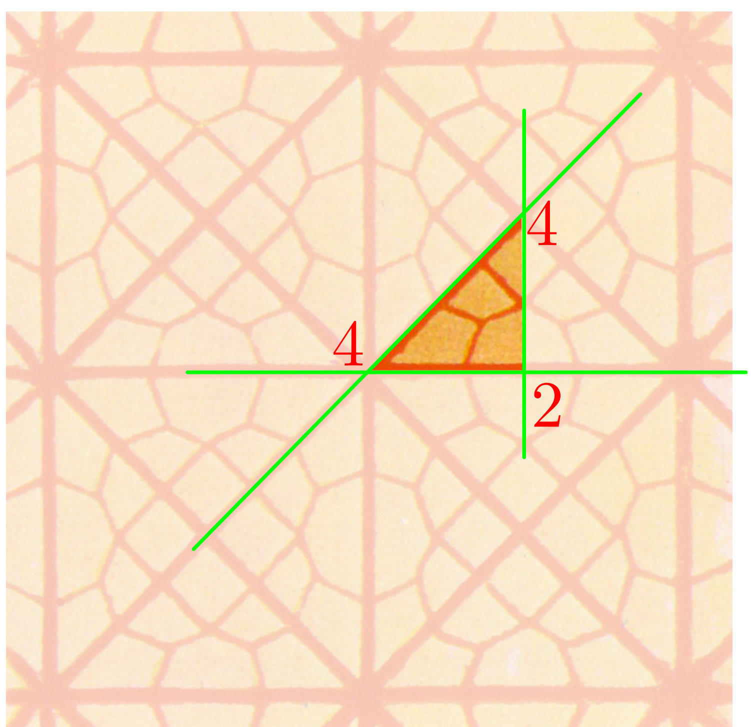


Let's now draw, in bright green, all the lines across which there is a reflection in the walllpaper group (we will sometimes call these lines *mirror lines* or *reflection lines*).

[10] Public Domain, https://commons.wikimedia.org/w/index.php?curid=3022620



In this example, we have an important feature, which is that the mirror lines bound a small region of the plane. We call this a *Kaleidoscope*. Let's focus in on one of these regions. We write in the number of mirror lines intersecting at each corner of the region:

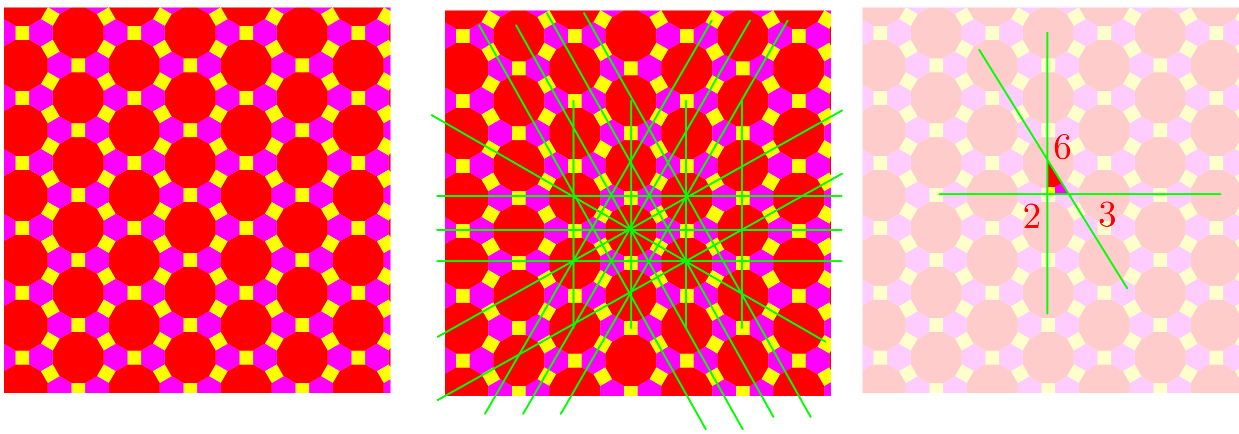


It turns out that all the information of the wallpaper pattern (and the wallpaper group) is contained in this kaleidoscope. In fact, you can take these three reflections and the triangular region of the pattern (which is called a *fundamental domain*) and recover the whole pattern.

We caputure the information in the *signature of the wallpaper pattern*, which in this case is **$*$442**.

The asterisk indicates the existence of a Kaleidoscope, and the numbers after indicate how many mirrors intersect at each vertex of the fundamental domain.

Here's another, slightly different, example, performing the same steps as above (except only drawing in some of the mirror lines in the second step):



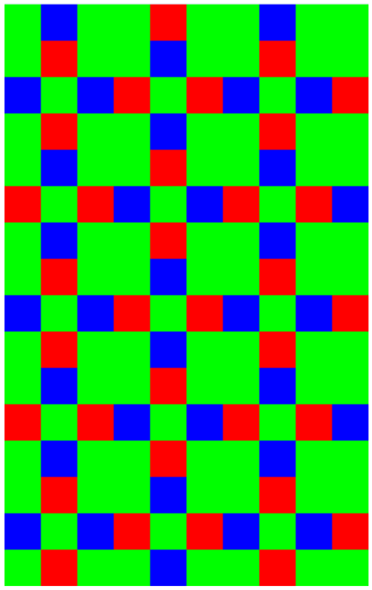
The signature of this wallpaper pattern is **$*$632**.

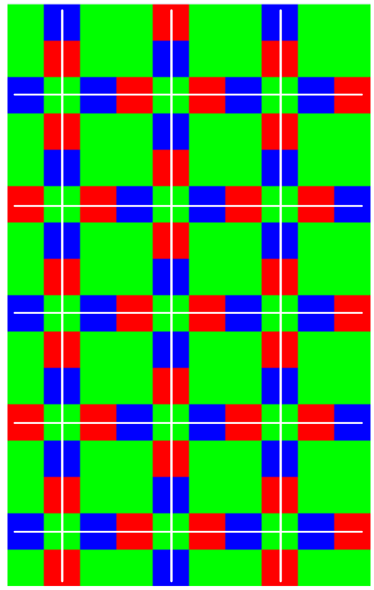
Feature 2 - Gyrations

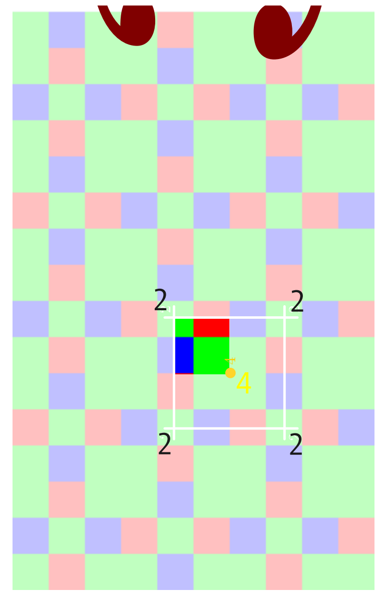
In the previous two examples, the reflections shown are *not* the only elements of the wallpaper group. For example, there are rotations centered at every point where two mirror lines intersect. However, each of these rotations can be obtained by composing two of the reflections across lines that meet at the rotation center.

When there is a rotation that does not arise in this way, we call it a *gyration*. This is illustrated in the following example[11]. We start by drawing in all the reflection lines and focusing in on one of the regions bounded by the reflection lines.

[11]By Ckendel at German Wikipedia - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=19113240







Here we notice that in the center of each square there is the center of a rotation *not* lying on a reflection line. Furthermore, the smallest rotation is by $\frac{\pi}{4}$, which if composed with itself four times is the identity. To record this information, we write down the number 4 *before* the $*$ in the signature.

There is one other thing to keep track of here. Each of the four Kaleidoscope points in the image on the right is mapped to the adjacent one under the action of the rotation. So in some sense these are all the same Kaleidoscope point, and we only want to keep track of one of them.

Definition. Two points in a wallpaper pattern are of the same type if there is a symmetry mapping one to the other.

Convention. Only one of each type of kaleidoscope and gyration point should be recorded in the signature.

So, the signature of this particular wallpaper pattern is **4∗2**.

The small square that appears with full colour in the third image is a fundamental domain for the wallpaper pattern. You can recover the whole wallpaper pattern by applying compositions of one of the reflections with the rotation to the fundamental domain.

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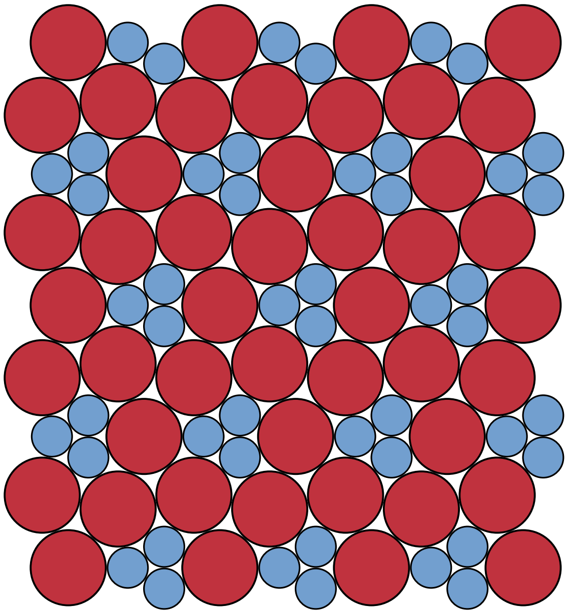
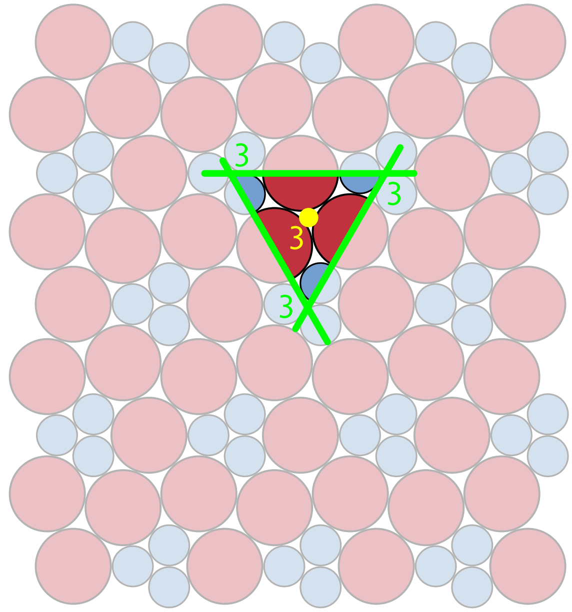
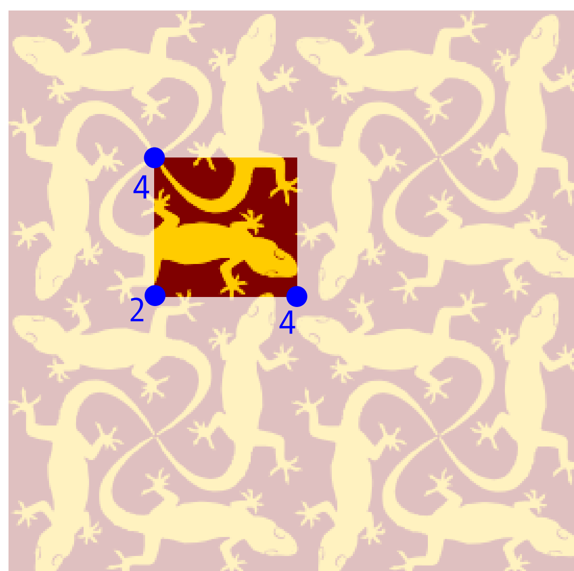
Here's another similar example[12], this one with signature **3∗3**.

[12] By Toby Hudson - Own work, following Tom Kennedy [2], [3], CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=10503540





It is possible that there are no reflections at all, in which case we move straight onto finding gyrations, keeping in mind we want to find all gyrations, but only one of each type. This kind of behaviour is exhibited in the following reptilian wallpaper pattern[13]:

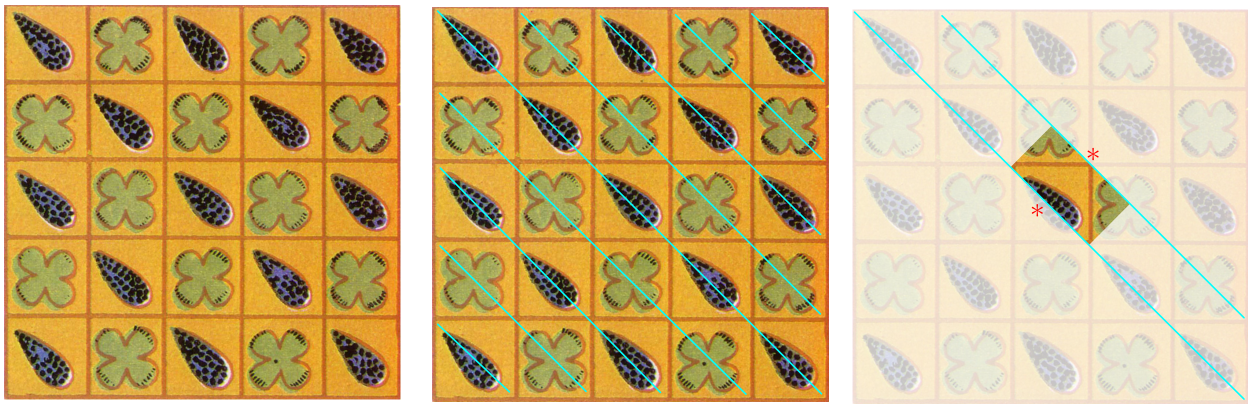


There are gyrations of order 4 and gyrations of order 2, but only three distinct types of rotation centers. The image on the right shows one of each type, and the full-strength square is a fundamental domain. The entire wallpaper pattern can be recovered by applying various compositions of the three rotations to the fundamental domain.

[13]By © Nevit Dilmen, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=28095443

It is worth noting that the top right-hand corner of the fundamental domain does not need to be indicated with a gyration number because it would be of the same type as the one on the bottom left. So, the signature of this wallpaper pattern is **442**.

There are a couple of strange things that can happen with only kaleidoscopes and gyrations which we haven't seen yet. Here is one example of the strangeness[14]:



The second image has all of the reflection lines drawn in. The new feature here is that all the reflection lines are parallel. However, in keeping with our desire to only have one of each type of feature, we are compelled to only include two of the reflections.

There is another feature present here which is not accounted for by the two reflections, and that is a translation in the direction parallel to the reflection lines. It turns out that unless the only features present are translations (as we will see when we introduce Feature 4 below), we ignore the translations when writing down our signature.

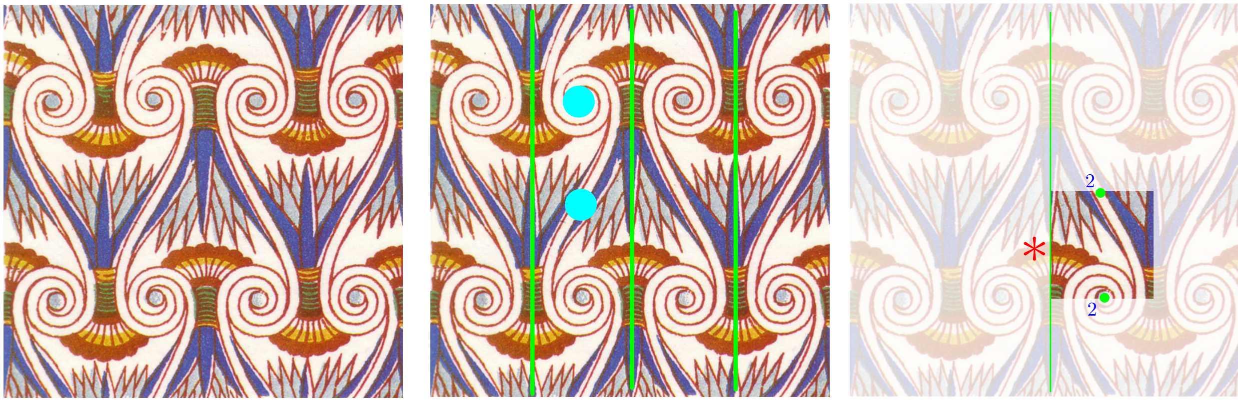
In the example, there are no gyrations, and we write down the signature as **, indicating two parallel reflections. The rightmost image also has a fundamental domain indicated. The two reflections and the translation (and the inverse of the translation) applied to the fundamental domain will recover the entire wallpaper pattern.

"But wait," you object, "there are glide reflections here, how come no one cares about any of the glide reflections!?"

A reasonable objection. Every glide reflection in this example, and the previous examples, can be written as compositions of the isometries that are accounted for by the features that we have written down in our signature (or in the case of the most recent example, the two reflections and the translation). There will be a time when we will need to pay attention to the glide reflections, but not yet! First another example[15] where reflection lines to not intersect.

[14]By Owen Jones - From en:The Grammar of Ornament (1856), by Owen Jones.Nineveh & Persia No 2 (plate 13), image 10, Public Domain, https://commons.wikimedia.org/w/index.php?curid=3021542

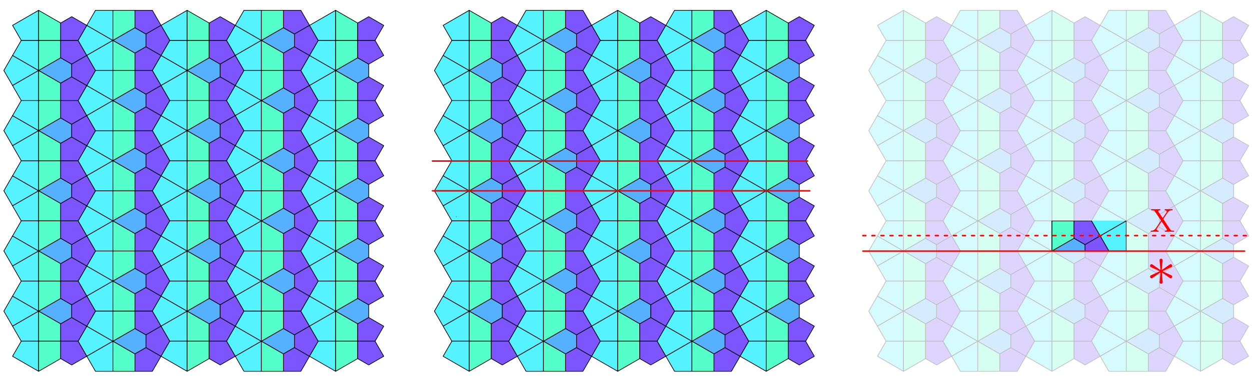
[15]Public Domain, https://commons.wikimedia.org/w/index.php?curid=3022561



As always, we start with all the reflection lines, followed by any gyrations with centers not on the reflection lines. In the middle image we have indicated all the reflections and one of each type of gyration, both of order 2. In this case, the gyrations take a reflection line to an adjacent reflection line, so there is only one type of reflection line. The third image indicates all the features we need, as well as a fundamental domain. The signature for this example is **22∗**. Remember, gyrations first, then kaleidoscopes.

Feature 3 - Glide Reflections

Okay, we're ready to see some wallpaper patterns which forbid us from ignoring glide reflections altogether. Here is the first[16]:



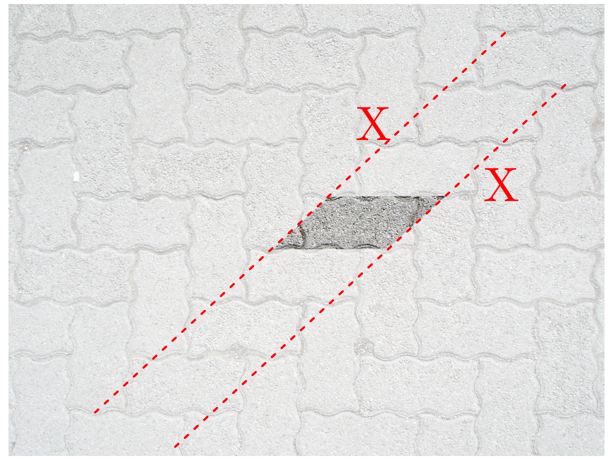
All the reflection lines are horizontal, there are no gyrations, and it appears (as in the middle picture) that there are two types of reflection lines. However, upon closer inspection, we see that there is a glide reflection, whose reflection axis (the dotted line in the rightmost image) runs right between two reflection lines, taking one to the other. So really, the two reflection lines are of the same type, and we only need to pay attention to one reflection line and one glide reflection. Every other element of the wallpaper group can be obtained by composing the reflection and glide reflection (and their inverses) shown in the right-most image (see if you can justify this to yourself).

So, denoting such a glide reflection by × we can write down the signature of this particular wallpaper pattern as ∗×.

[16]By Harry Princeton - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=138742085

There are two important rules when dealing with glide reflections in signatures. The first is that the $\times$ always appears at the end of the signature (not really an important rule, but a typographical convention). The second is that we only include glide reflections when the reflection axis of the glide reflection does not intersect a reflection line (which includes coinciding with a reflection line).

Some more![17]



This one is tricky! There are no reflection lines, and no gyrations (pay attention to the squiggles on the bricks). However, there are two parallel glide reflections, neither of the same type. To see they're not of the same type, look at the curve on the short ends of the bricks that intersect each glide reflection axis. In one axis, the curves go away from the brick. In the other, they move towards the brick. Tricky!

In this case, the signature is $\times\times$. We can apply the two glide reflections (and their inverses, and all compositions) to the fundamental domain indicated on the right-most image to recover the wallpaper pattern.

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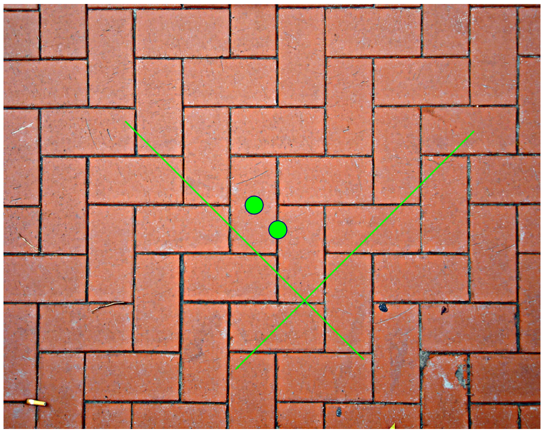
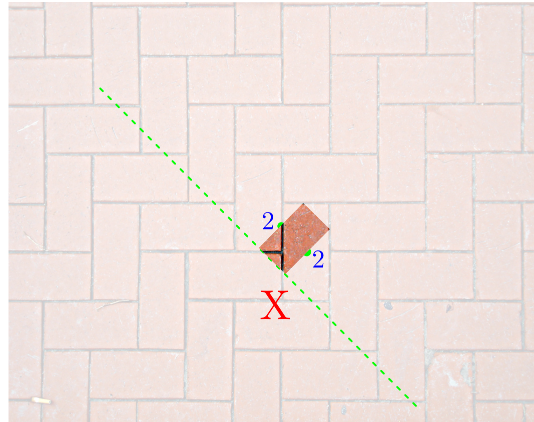
These glide reflections are tricky, so let's see one more in the form of a herringbone brick pattern[18]:

[17]By Dmharvey at English Wikipedia - Transferred from en.wikipedia to Commons., Public Domain, https://commons.wikimedia.org/w/index.php?curid=37301029

[18]By Dmharvey at English Wikipedia - Transferred from en.wikipedia to Commons., Public Domain, https://commons.wikimedia.org/w/index.php?curid=3021553

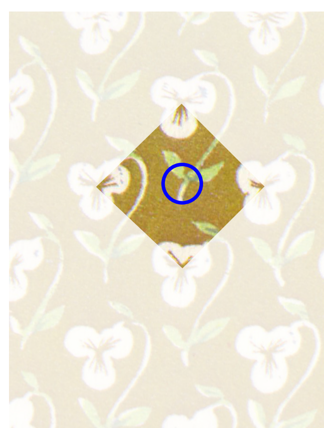
 

This is similar to the previous example except the bricks do not have those pesky curvy edges. In this case, there are no reflections, but there are two distinct types of gyration, each of order 2. There also appear, at first glance at least, to be many different glide reflection axes, parallel to one of the two indicated on the image on the left.

Amazingly, all of the glide reflections turn out to be of the same type, and can be obtained by various compositions of just a single glide reflection (and its inverse) and one of the gyrations. The image on the right shows one of these glide reflections, the two gyrations, and a fundamental domain. The signature is **22×**.

Feature 4 - Translations

If you have none of the above features, the only symmetries must be translations, and two translations in linearly independent directions will exist (the existence of two linearly independent translations is one way to define what a wallpaper pattern is, which we didn't do very formally). For such a wallpaper pattern (an example of which follows this paragraph[19]), its signature is simply **O**.

Note that this is the only kind of wallpaper pattern for which we indicate translations in the signature.

[19]By Owen Jones - en:Image:Wallpaper_group-p1-3.jpg, Public Domain, https://commons.wikimedia.org/w/index.php?curid=2979174

It's worth noting that we have dealt with reflections, rotations, glide reflections, and translations, which are all the possible types of isometries! Although what I've said doesn't pass as a proof, it is true that there are no other types of features that can appear in the signature of a wallpaper pattern.

4.2. The Magic Theorem

We've seen a whole bunch of different wallpaper patterns, and different signatures that can appear. It seems that anything can happen! However, if you try to construct a wallpaper pattern with signature, say $*832$ you will have a really hard time (because it turns out such a signature does not exist).

The main theorem governing which signatures can and can't exist will be referred to as *The Magic Theorem* and will not be proved in this course. The Magic Theorem involves counting a value, called the *cost*, associated to each symbol that appears in the signature of a wallpaper pattern.

Every symbol corresponds in some way to an isometry of $\mathbb{R}^2$, so we can group them into whether or not they are orientation-preserving or orientation-reversing symbols. The symbol **O** corresponds to translations, which are orientation-preserving. All numbers that appear before a $*$ (if there is one at all) correspond to rotations, so they are also orientation-preserving. The symbols $*$, $\times$, and any numbers appearing after a $*$ correspond to reflections or glide reflections, and are thus orientation-reversing.

Orientation-preserving		**Orientation-reversing**	
Symbol	Cost	Symbol	Cost
O	2	$*$ or $\times$	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{2}{3}$	3	$\frac{1}{3}$
4	$\frac{3}{4}$	4	$\frac{3}{8}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{n-1}{n}$	n	$\frac{n-1}{2n}$

Let's compute the cost of all the signatures we computed.

Signature	Cost
$*442$	$1+\frac{3}{8}+\frac{3}{8}+\frac{1}{4}=2$
$*632$	$1+\frac{5}{12}+\frac{1}{3}+\frac{1}{4}=2$
$4*2$	$\frac{3}{4}+1+\frac{1}{4}=2$
$3*3$	$\frac{2}{3}+1+\frac{1}{3}=2$
442	$\frac{3}{4}+\frac{3}{4}+\frac{1}{2}=2$
$**$	$1+1=2$
$22*$	$\frac{1}{2}+\frac{1}{2}+1=2$
$*\times$	$1+1=2$
$\times\times$	$1+1=2$

Signature	Cost
22×	$\frac{1}{2} + \frac{1}{2} + 1 = 2$
O	2

Notice anything? What you notice is the content of The Magic Theorem, whose beautiful proof is sadly beyond the scope of this course.

Theorem 18 (*The Magic Theorem*). The total cost of the signature of any wallpaper pattern is 2.

Lecture 19 - 06/24

The Magic Theorem is not only magic, but it also helps us correctly identify signatures. For example, if we find two gyrations of order 2 and there are no reflections, then we must have missed something since the total cost is only 1. Furthermore, we know that since there is no $*$ in our signature, the extra cost must come from either two more gyrations or a glide reflection.

Before moving on, let's take a moment to summarise the steps we take to find the signature of a wallpaper pattern.

How to find the signature of a wallpaper pattern:

1. Draw in all reflections. If there are any reflections at all, the signature gets a $*$. If there is a bounded region, you only need to pay attention to that region in what follows.
2. Find gyrations not on reflection lines.
3. Look for glide reflections with axes that do not intersect reflection lines. If there are two parallel glide reflections, the signature is ××.
4. None of the above? There must be two linearly independent translations and the signature is **O**.
5. Make sure there is only one type of each feature, and that the total cost of the signature is 2.

The goal is to now classify, with the help of this powerful theorem, all the possible signatures of wallpaper patterns.

Proposition 19. There are 17 possible wallpaper patterns.

Proof. Case 1: If an O appears in the signature, then the signature must be **O**.

Case 2: The signature consists only of gyrations. Since the cost of any gyration is less than one, there must be at least three gyrations in the signature. Since the cost of a gyration is at least $\frac{1}{2}$, there must be at most four gyrations in the signature.
If there are exactly three gyrations, the mean cost of the symbols is $\frac{2}{3}$. If there are no 2s in the signature, then **333** is the only possibility. If there is a 2 in the signature, then the mean cost of the other two symbols is $\frac{3}{4}$. The only possibilities in this case are **442** and **632**.
If there are exactly four gyrations, the mean cost of the symbols is $\frac{1}{2}$, implying that the only possible signature is **2222**.

Case 3: The signature contains only kaleidoscopes. That is, the signature is of the form

$$*n_1 n_2 n_3 \cdots n_k.$$

Such a signature has cost 2 if and only if

$$
\begin{aligned}
& \frac{n_1-1}{2n_1}+\cdots+\frac{n_k-1}{2n_k}=1 \\
\iff \qquad & \frac{n_1-1}{n_1}+\cdots+\frac{n_k-1}{n_k}=2
\end{aligned}
$$

which is true if and only if the cost of the signature $\boldsymbol{n_1 n_2 \cdots n_k}$ is 2. Since we already know which signatures consisting only of gyrations have cost 2, we can conclude that the only possible signatures in this case are **∗632**, **∗442**, **∗333**, and **∗2222**.

Case 4: There are at least two non-numerical symbols. The only possibilities in this case are ∗∗, ∗×, and ××.

Case 5: There are gyrations and exactly one symbol. The only possibilities here are **22∗**, and **22×**

Case 6: There are gyrations and kaleidoscopes. It is an exercise for you to show that the only possibilities here are **2∗22**, **4∗2**, and **3∗3**.

This completes every possible case and completes the proof (since we have listed the 17 possible signatures). ∎

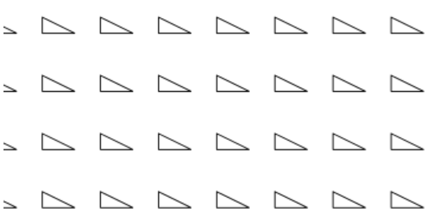
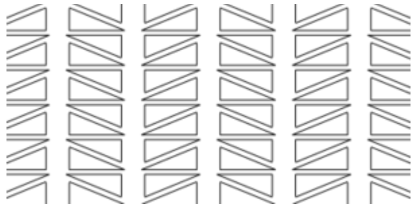
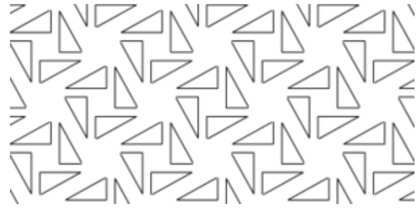
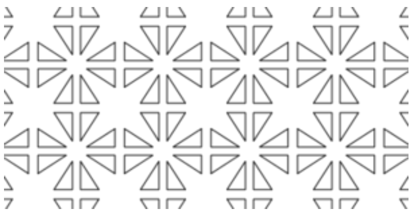
Exercise. Convince yourself that the six cases in the proof cover all possibilities.

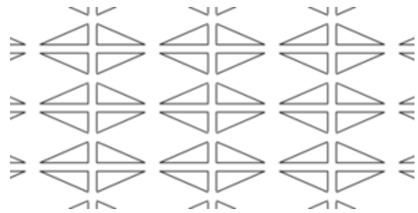
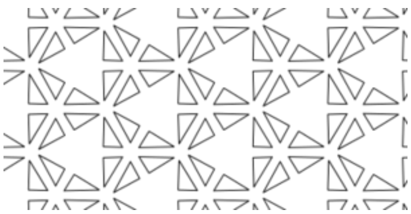
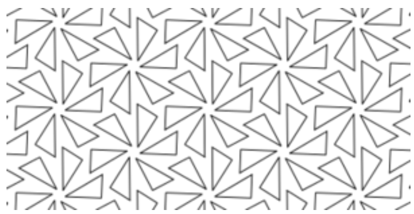
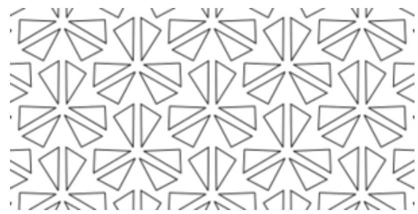
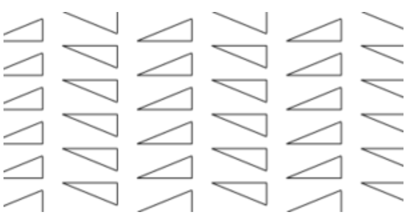
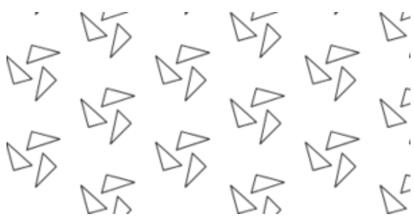
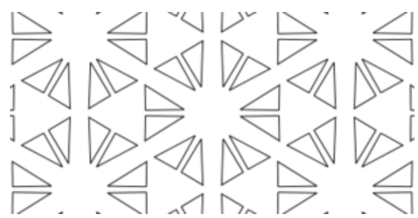
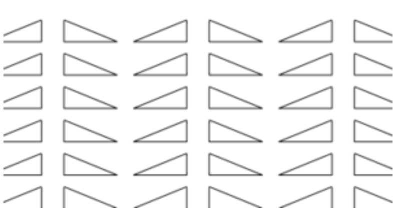
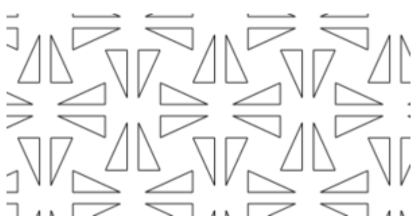
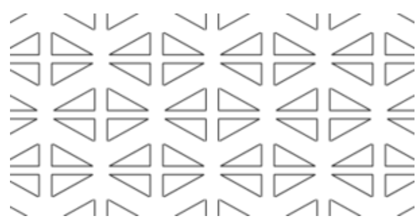
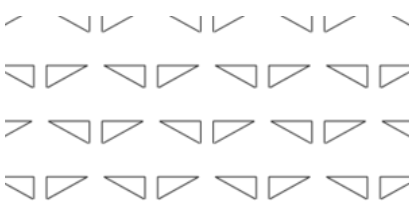
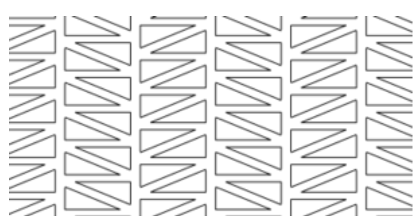
Here are a couple of cool websites that are related to wallpaper patterns:

- https://www.geometrygames.org/KaleidoPaint/
- https://math.hws.edu/eck/js/symmetry/wallpaper.html

Although the proof above tells us that there are only 17 possible signatures, it does not tell us anything about whether or not all 17 signatures actually occur. It turns out that they do! The next page has 17 wallpaper patterns[20], one for each signature. Can you match them up with their signatures?

[20] https://eschermath.org/wiki/Wallpaper_Patterns.html

Lecture 20 - 06/26

4.3. Tilings

In this section, all tilings of the plane will be by regular polygons, and tiles will be placed so vertices meet at the same point in the plane.

There are three *regular* tilings of the plane. That is, where each tile is the same regular polygon. These are by squares, triangles, and hexagons. The reason we can do this is because the interior angle of a square, triangle, and hexagon are $\frac{2\pi}{4}$, $\frac{2\pi}{6}$, and $\frac{2\pi}{3}$ respectively. Each of these angles is of the form $\frac{1}{k}2\pi$ for some positive integer k, and hence the tiles fit snugly around each vertex, filling out all the space surrounding the vertex.

Exercise. Show that these are the only three regular tilings of the plane.

Definition. A tiling of the plane is *semi-regular* if all tiles are regular polygons, and every vertex is of the same type.

Recall that every vertex being of the same type means that for any two vertices, there is a symmetry of the tiling (that is an isometry that leaves the tiling unchanged) that maps one vertex to the other.

The question we wish to investigate is what are all the possible semi-regular tilings of the plane. To approach this question, imagine the situation around a particular vertex in a semi-regular tiling. The first thing to notice is that if k polygons meet at one vertex, then k polygons meet at every vertex (since every vertex is of the same type). So, suppose the k polygons had $n_1, n_2, \dots, n_k$ sides. The interior angles of the k polygons must add up to 2π.

Now, the interior angle of a regular n-gon is $\frac{n-2}{n}\pi$ (this is an exercise for you to prove). So, the numbers $n_1, n_2, \dots, n_k$ must satisfy

$$\frac{n_1-2}{n_1} + \frac{n_2-2}{n_2} + \dots + \frac{n_k-2}{n_k} = 2.$$

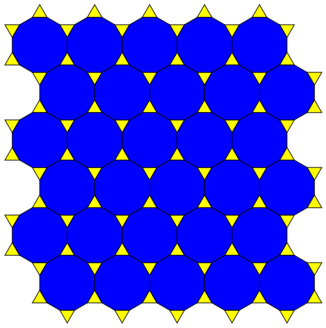
The possible positive integers (greater than 2, since we are talking about polygons) that satisfy this are:

$$\begin{array}{ll} 3,12,12 & 3,4,4,6 \\ 4,8,8 & 4,4,4,4 \\ 4,6,12 & 3,3,3,4,4 \\ 6,6,6 & 3,3,3,3,6 \\ 3,3,6,6 & 3,3,3,3,3,3. \end{array}$$

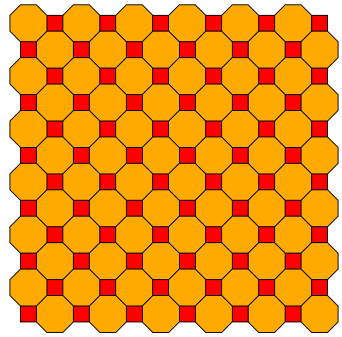
Exercise. Prove that these are the only 10 lists of positive integers (≥ 3) satisfying the required conditions.

The lists $6,6,6$, $4,4,4,4$, and $3,3,3,3,3,3$ correspond to the regular tilings by hexagons, squares, and triangles. The other seven lists correspond to the eight semi-regular tilings shown below[21] (with $3,3,3,4,4$ giving rise to two distinct tilings).

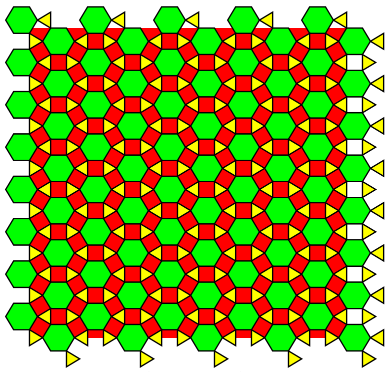
[21] https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons#Archimedean,_uniform_or_semiregular_tilings



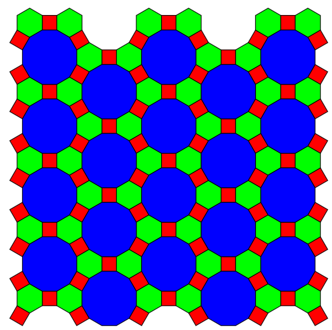
3, 12, 12



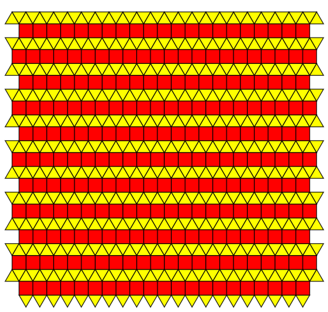
4, 8, 8



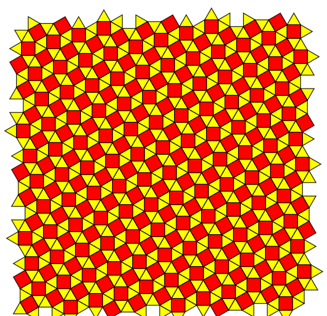
3, 4, 4, 6



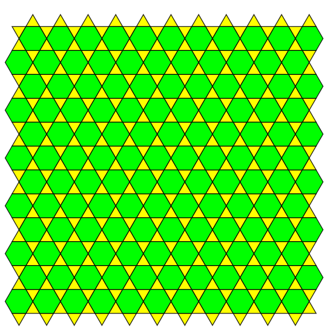
4, 6, 12



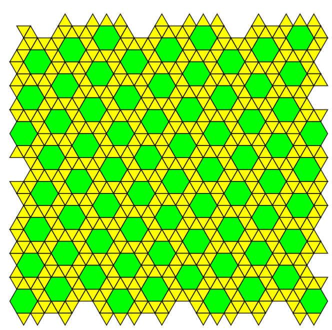
3, 3, 3, 4, 4



3, 3, 3, 4, 4



3, 3, 6, 6



3, 3, 3, 3, 6

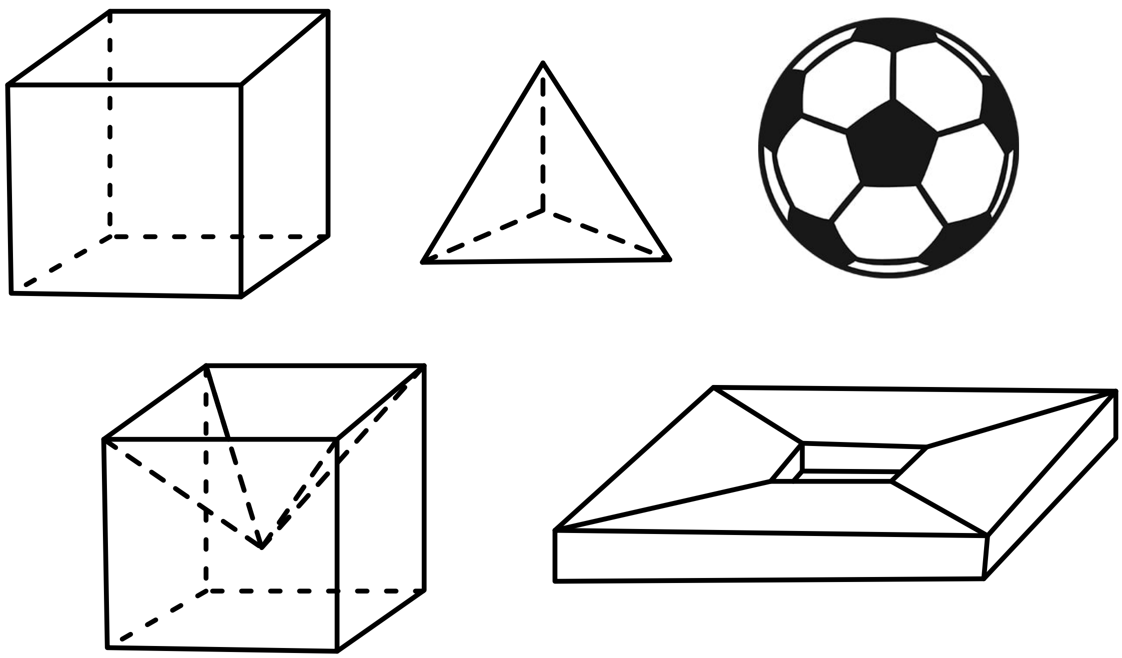
Lecture 21 - 06/29

5. Polytopes

A line segment is a 1-dimensional polytope. A bunch of line segments meeting at their end points forms a polygon, which is a 2-dimensional polytope. A bunch of polygones meeting at their edges is a polyhedron, which is a 3-dimensional polytope. In general, such objects are called polytopes. Let's start with the polyhedra.

5.1. Polyhedra

For us, a polyhedra will be a 3-dimensional object formed by glueing together polygons (not necessarily regular, and not necessarily convex) along their edges. In our polyhedra, edges of polygons must coincide with edges of polygons, and vertices with vertices. Here are some examples of polyhedra.



We have, in reading order, a *cube*, a *tetrahedron*, a *soccer ball* otherwise known as a *truncated icosahedron*, a polyhedron without a name, and a *blocky donut* or a *krispy kremehedron*.

We can see that our polyhedra don't have to be convex (a shape is *convex* if the line segment containing any two points in the shape is also in the shape), and they can even have holes in them! However, the faces cannot have holes (which is why the top face of the blocky donut had to be cut into four faces). The faces have to be polygons, but the polygons do not have to be regular!

A plethora of examples of polyhedra can be found at this wonderful website.

One (of the many) interesting things about polyhedra is that when a bunch of polygons meet at a vertex, the sum of the internal angles of the polygons that meet at that vertex *do not* necessarily sum to 2π, like they do when polygons tile the plane.

Definition. The *angle defect* at a vertex is $2\pi - S$ where S is the sum of the internal angles of the polygons that meet at the vertex.

Take, for example, a vertex of a cube. Here, three squares meet, and each of the angles of these squares is $\frac{\pi}{2}$. Therefore, the angle defect at each vertex of a cube is $2\pi - 3(\frac{\pi}{2}) = \frac{\pi}{2}$. On a soccer ball, the angle defect of each vertex is $\frac{\pi}{15}$.

The angle defect of a vertex does not necessarily have to be positive. The eight vertices at the hole of the krispy kremehedron each has an angle defect of $\frac{-\pi}{2}$ (assuming that the interior angles of the each of the 4-gons on the top are $\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, and $\frac{3\pi}{4}$).

Another property of polyhedra worth paying attention to is the number of vertices, edges, and faces, which we generally denote by V, E, and F.

For a cube, $V = 8$, $E = 12$, and $F = 6$.

Let's organise what we've got so far. To do this, we will consider the quantity $V - E + F$, and instead of angle defects for individual vertices, we will take the sum of all angle defects.

Let's compute the cost of all the signatures we computed.

Polyhedron	**Total angle defect**	$\boldsymbol{V - E + F}$
Cube	4π	$8 - 12 + 6 = 2$
Tetrahedron	4π	$4 - 6 + 4 = 2$
Soccer ball	4π	$60 - 90 + 32 = 2$
Unnamed thingie	4π	$9 - 16 + 9 = 2$
Krispy kremehedron	0	$16 - 32 + 16 = 0$

This table suggests a relationship between the total angle defect of a polyhedron and the quantity $V - E + F$. And indeed there is one!

Theorem 20 (Descartes' formula). Let T be the total angle defect of a polyhedron. Let V, E, and F be the number of vertices, edges, and faces of the polyhedron respectively. Then

$$T = 2\pi(V - E + F).$$

Lecture 22 - 07/03

Proof (of Descartes' formula). This proof will rely on the following fact (left as an exercise for you to prove) about polygons: The sum of the interior angles of an n-gon is $(n-2)\pi$.

The quantity $2\pi(V - E + F)$ can be thought of as arising from giving each vertex a value of 2π, each edge a value of -2π, and each face a value of 2π, and then summing up all the values over the entire polyhedron.

The idea of the proof is to move these values around, without changing the total sum, until we are left with the sum of all the angle defects. We do this in two steps.

1. On each edge, we distribute $-\pi$ to each of the two adjacent faces. This leaves a value of 0 on each edge, and a value of $2\pi - n\pi$ on each face, where n is the number of edges on that face. The value of each vertex remains at 2π.
2. Several interior angles of faces meet at each vertex. So, for each vertex, distribute the value of the interior angle to the corresponding face.

After Step 2, each vertex is left with a value of the angle defect of that vertex, each edge still has value 0, and each face now has a value of $2\pi - n\pi$ plus the sum of the interior angles of the face, which is equal to $2\pi - 2\pi$. Therefore, the total value remaining on the polyhedron is the sum of the angle defects, completing the proof. ∎

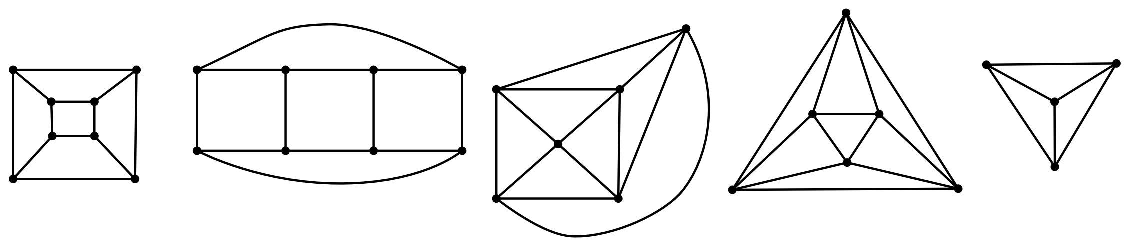
Definition. A polyhedron is *spherical* if it has no holes (more technically, if it is homeomorphic to a sphere).

Roughly, a spherical polyhdron is one which if inflated, becomes a sphere! So, for example, a cube and tetrahedron are spherical, but the Krispy kremehedron is not.

To each polyhedron we can create a corresponding *graph* (that is, a bunch of vertices denoted by dots and edges denoted by lines joining vertices). The graph is precisely that created by taking the vertices and the edges of the polyhedron! Here is a great fact about spherical polyhedra.

Fact 21. The graph formed by the vertices and edges of a spherical polyhedron can be embedded in $\mathbb{R}^2$.

In graph theory terminology, every graph arising from a spherical polyhdron is a planar graph. Here are some examples of viewing spherical polyhdra as planar graphs. Note that in each of these presentations of a polyhedron, one of the faces is around the outside! From left to right we have a cube, a cube, an octahedron, an octahedron, and a tetrahedron:



The reason we are talking about these graphs is to prove Euler's formula.

Theorem 22 (Euler's Formula). For a spherical polyhedra, $V - E + F = 2$.

Proof. Since every spherical polyhedron corresponds to a connected planar graph, it is enough to prove that the formula holds for all connected planar graphs. The proof will proceed by induction on the number of edges.

A connected planar graph with zero edges is just a single vertex. In this case

$$V - E + F = 1 - 0 + 1 = 2$$

which proves the base case.

Now, suppose there is an edge e. If e connects two distinct vertices, contract it (that is delete the edge and make both vertices at the end of the edge the same vertex). Contracting an edge reduces V and E by one and leaves F unchanged. So, $V - E + F$ is unchanged.

If e is a loop, delete it! This reduces F and E by one each, but leaves V unchanged. Therefore, $V - E + F$ is unchanged.

Alas, the result follows by induction. ∎

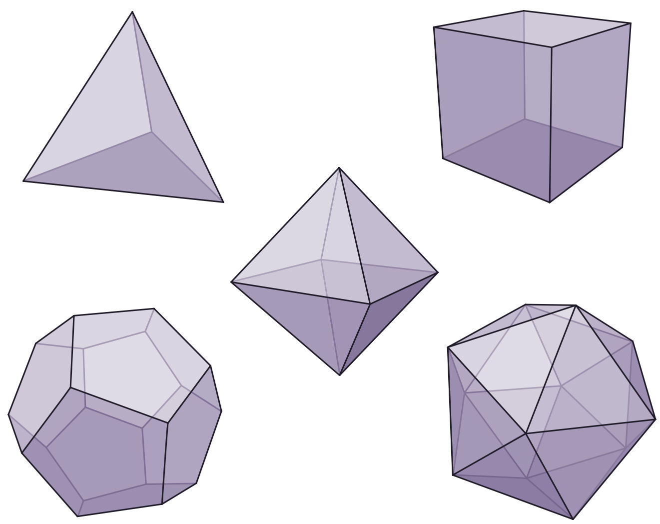
So, now we know that the sum of the angle defects of a spherical polyhedron is always 4π. Neat.

5.2. The Five Platonic Solids

Some of the polyhedrons we have encountered so far are particularly nice. The tetrahedron and cube both have the property that every face is the same regular n-gon, and the same number of faces meet at each vertex. Such a polyhedron is called regular.

Definition. A *regular polyhedron* is a spherical polyhedron constructed so that every face is the same regular n-gon, and the same number of faces meet at each vertex.

It turns out that there are only five regular polyhedron, which are called the platonic solids. Here they are[22].



In the top-left is a *tetrahedron*, top-right is a *cube*, bottom-left is a *dodecahedron*, bottom-right is an *icosahedron*, and an *octahedron* in the middle. Here's some notation for platonic solids.

Lecture 23 - 07/06

Definition. If q p-gons meet at every vertex of a regular polyhedron we denote the polyhedron by the Schläfli symbol $\{p, q\}$.

Here is a table organising the information for the five platonic solids.

Platonic solid	**At each vertex...**	**Schläfli symbol**
Tetrahedron	Three triangles	$\{3, 3\}$
Cube	Three squares	$\{4, 3\}$
Octahedron	Four triangles	$\{3, 4\}$
Icosahedron	Five triangles	$\{3, 5\}$
Dodecahedron	Three pentagons	$\{5, 3\}$

[22]Drummyfish, CC0, via Wikimedia Commons

The goal now is to prove that these are precisely the platonic solids that can exist. We'll first show that the Schläfli symbols above are the only possible ones, and then we will show that the five platonic solids listed above actually exist.

Proposition 23. There are only five possible regular polyhedra.

Proof. Suppose a regular polyhedron has Schläfli symbol $\{p, q\}$. By Descartes' formula, the angle defect at each vertex must be positive (since they are all the same and the sum of all of them is 4π). So,

$$\begin{aligned} & q\left(1-\frac{2}{p}\right)\pi < 2\pi \\ \iff \quad & 1-\frac{2}{p} < \frac{2}{q} \\ \iff \quad & \frac{1}{p}+\frac{1}{q} > \frac{1}{2} \\ \iff \quad & (p-2)(q-2) < 4 \end{aligned}$$

Since $p, q \geq 3$, the only possibilities are $\{p, q\} = \{3,3\}, \{3,4\}, \{4,3\}, \{3,5\}, \{5,3\}$. ■

Now, although we can draw pictures of them, it doesn't mean they must exist. Pictures can be misleading! However, to prove the solids exist we can provide coordinates for their vertices, which will provide us with a way of checking the faces are indeed regular polygons.

Cube ($\{4,3\}$). A cube exists with the eight vertices at

$$\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3 : \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}\}.$$

Exercise. Find the six four-element subsets of the set of vertices of the cube that give the vertices of the six faces.

Octahedron ($\{3,4\}$). Given a cube, we can construct an octahedron by placing a vertex in the center of each face of the cube. An octahedron exists with vertices at

$$\{e_1, -e_1, e_2, -e_2, e_3, -e_3\} \subset \mathbb{R}^3$$

where e_i is the ith standard basis vector (so for example, $e_2 = (0,1,0)$).

Exercise. Prove these are indeed the vertices of a regular polyhedron.

Constructing a polyhedron from another this way is called taking the dual.

Definition. The *dual* S of a regular polyhedron T is the regular polyhedron constructed by placing a vertex of S in the centre of every face of T, and connecting two vertices of S by an edge if the two faces of T share an edge. A set of vertices of S form the vertices of a face if the corresponding faces of T all meet at a vertex of T.

Exercise. Convince yourself that the cube and octahedron are duals of each other. Compare their Schläfli symbols. What do you notice?

Lecture 24 - 07/08, Quiz

Lecture 25 - 07/10, I've clearly misnumbered my lectures

Lecture 26 - 07/13

Tetrahedron ($\{3,3\}$). Writing down the vertices of a regular tetrahedron is surprisingly tricky. However, we can use the vertices of a cube to build a tetrahedron. A tetrahedron exists and has vertices

$$\{(1,1,1),(-1,-1,1),(-1,1,-1),(1,-1,-1)\}.$$

The dual of a tetrahedron ($\{3,3\}$) is a regular polyhedron with Schläfli symbol $\{3,3\}$, which is a tetrahedron!

Icosahedron ($\{3,5\}$). For this one, set $\varphi = \frac{1+\sqrt{5}}{2}$ (more commonly known as the golden ratio). Then an icosahedron can be constructed with vertices that come in groups of four, where each group forms a rectangle with side lengths 2 and 2φ in each of the three coordinate planes:

$$\{(0,\varepsilon_1,\varepsilon_2\varphi) : \varepsilon_1,\varepsilon_2 \in \{-1,1\}\} \cup \{(\varepsilon_1,\varepsilon_2\varphi,0) : \varepsilon_1,\varepsilon_2 \in \{-1,1\}\} \cup \{(\varepsilon_1\varphi,0,\varepsilon_1) : \varepsilon_1,\varepsilon_2 \in \{-1,1\}\}.$$

Dodecahedron ($\{5,3\}$). We can coordinatise the dodecahedron in a similar way. In this case, we split up the vertices into four groups. One of which forms the vertices of a cube. The other three groups consist of four vertices, each defining a rectangle with side lenghts 2φ and $\frac{2}{\varphi}$ in a coordinate plane. Define the four sets

$$R_{\{xy\}} = \left\{\left(\frac{\varepsilon_1}{\varphi},\varepsilon_2\varphi,0\right) : \varepsilon_1,\varepsilon_2 \in \{-1,1\}\right\}$$
$$R_{\{yz\}} = \left\{\left(0,\frac{\varepsilon_1}{\varphi},\varepsilon_2\varphi\right) : \varepsilon_1,\varepsilon_2 \in \{-1,1\}\right\}$$
$$R_{\{z,x\}} = \left\{\left(\varepsilon_2\varphi,0,\frac{\varepsilon_1}{\varphi}\right) : \varepsilon_1,\varepsilon_2 \in \{-1,1\}\right\}$$

and C, where C is the set of eight vertices of the cube coordinatised as above. Then the vertices of a dodecahedron are given by $R_{\{xy\}} \cup R_{\{yz\}} \cup R_{\{z,x\}} \cup C$.

Exercise. Check that these sets of vertices to indeed define the vertices of regular polyhedra.

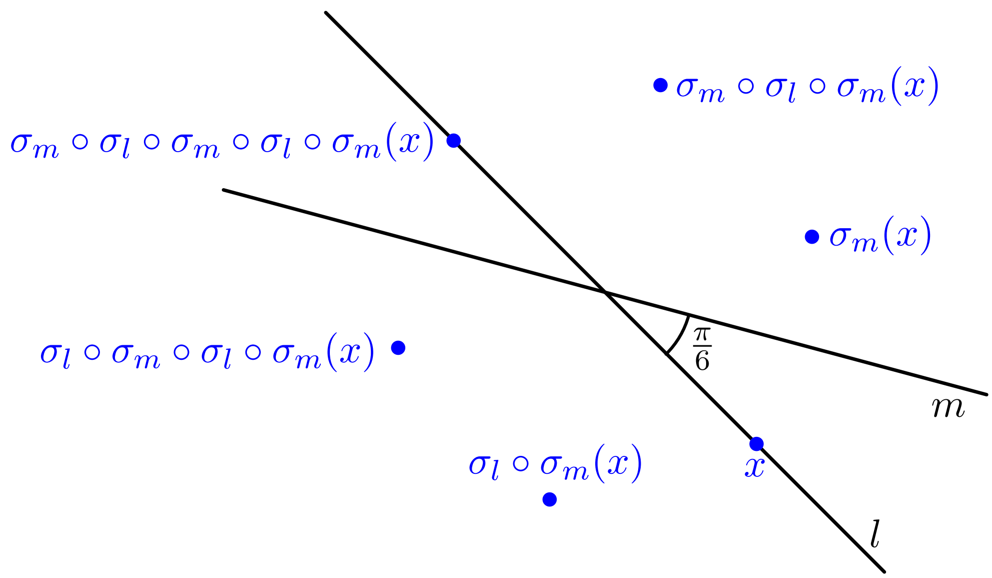
It turns out that the dual of an icosahedron is a dodecahedron (and vice versa).

5.3. Constructing polyhedra via reflections

One way to construct polyhedra is to do what we did above, and simply write down coordinates. Another way is to build it up bit by bit by using symmetries. The guiding principle we will keep in mind is that if we want a symmetric object, build it out of the symmetries!

Let's start with the following example. Suppose l and m are two lines in $\mathbb{R}^2$ that intersect at an angle of $\frac{\pi}{6}$. Let x be a point of l not on m. Now let's reflect the point around as much as we can using reflections σ_l and σ_m across l and m. The diagram below shows the set of six points

$$\{x,\sigma_m(x),\sigma_l\circ\sigma_m(x),\sigma_m\circ\sigma_l\circ\sigma_l(x),\sigma_l\circ\sigma_m\circ\sigma_l\circ\sigma_l(x),\sigma_m\circ\sigma_l\circ\sigma_m\circ\sigma_l\circ\sigma_l(x)\}.$$



These six points are all the points you can get from x by applying compositions of σ_l and σ_m (you can check this by taking any of the six points, applying one of σ_l or σ_m, and making sure you get one of the six points already obtained).

The best part about these six points is that they form the vertices of a regular hexagon!

Exercise.
- What do you get if x is not on l or m?
- Suppose now that l and m meet at an angle of $\frac{\pi}{n}$. Following the same procedure as above, how many points do you obtain? Show that these points form the vertices of a regular k-gon for some k.
- Suppose now that l and m meet at an angle of θ. What values of θ guarantee that x only gets sent to finitely many points under compositions of σ_m and σ_l?

The goal now is to do something similar for polyhedra. We're going to ramp everything up one dimension. In order to do this, we need to have a notion of angle between two planes in $\mathbb{R}^3$.

Definition. The *dihedral angle* between two planes in $\mathbb{R}^3$ is the angle between their normal vectors.

There are many choices for the angle between two vectors. Unless otherwise stated, the dihedral angle θ between the two planes is taken to lie in the range $0 \le \theta \le \pi$.

Just like in $\mathbb{R}^2$, we can reflect across a plane in $\mathbb{R}^3$.

Definition. Let P be a plane in $\mathbb{R}^3$. Define the *reflection across* P, denoted σ_P as follows. For $x \in \mathbb{R}^3$, $\sigma_{P(x)}$ is the unique point in $\mathbb{R}^3$ satisfying
- The midpoint $\frac{1}{2}\left(\sigma_{P(x)} + x\right)$ is on P, and
- $2\,\mathrm{dist}\left(\sigma_{P(x)}, P\right) = \mathrm{dist}\left(\sigma_{P(x)}, x\right)$.

Exercise. Prove this definition agrees with the one given earlier in the course for the reflection across a line.

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Alright, let's try to imitate the construction of the hexagon above, but this time with three planes in $\mathbb{R}^3$, and our goal is to construct one of the platonic solids.

Consider the planes

$$P_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$$
$$P_2 = \{(x, y, z) \in \mathbb{R}^3 : x = y\}$$
$$P_3 = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$$

The planes have normal vectors

$$n_1 = (1, 0, 0), \quad n_2 = (1, -1, 0), \quad \text{and} \quad n_3 = (0, 1, -1)$$

respectively. Then, if we denote by θ_{ij} the dihedral angle between P_i and P_j we get

$$\theta_{12} = \frac{\pi}{4}, \quad \theta_{13} = \frac{\pi}{2}, \quad \text{and} \quad \theta_{23} = \frac{\pi}{3}.$$

If we let p be the point $(1, 1, 1)$ (which lies on P_2 and P_3 but not on P_1) and consider where p gets sent under all compositions of $\sigma_{P_1}, \sigma_{P_2}$, and σ_{P_3} we get the eight points

$$\{(1,1,1), (-1,1,1), (1,-1,1), (1,1,-1), (-1,-1,1), (-1,1,-1), (1,-1,-1), (-1,-1,-1)\},$$

which happen to be the vertices of a cube! If we do the same thing but this time starting with the point $(0, 0, 1)$ (which lies on P_1 and P_2, but not P_3) we get the six points

$$\{(0,0,1), (0,0,-1), (0,1,0), (0,-1,0), (1,0,0), (-1,0,0)\}.$$

The vertices of an octahedron! For a visual of this process, visit this Desmos graph.

Exercise. Let P_1 and P_2 be two planes intersecting at a dihedral angle of θ at the line l. Show that $\sigma_{P_1} \circ \sigma_{P_2}$ is a rotation about l by an angle of 2θ.

Why do the particular dihedral angles of $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ give us the cube and octahedron? Part of the answer is that the Schläfli symbols are $\{3, 4\}$ and $\{4, 3\}$. Around a face of a cube (or a vertex of an octahedron), there is an order-4 rotation. This rotation is obtained by the composition of the two reflections across the planes that have a dihedral angle of $\frac{\pi}{4}$. Around the vertex of a cube (or a face of an octahedron), there is an order-3 rotation, accounted for by the dihedral angle $\frac{\pi}{3}$. Finally, around the center of an edge there is an order-2 rotation, corresponding to the $\frac{\pi}{2}$ dihedral angle.

Here's the take-home message. In $\mathbb{R}^2$, the existence of two lines that intersect at $\frac{\pi}{n}$ corresponds to the existence of a regular n-gon. In $\mathbb{R}^3$, the existence of three planes with dihedral angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ corresponds to the existence of regular polyhedra with Schläfli symbols $\{3, 4\}$ and $\{4, 3\}$.

This leads to a very natural question. Can we play the same game for the tetrahedron, icosahedron, and dodecahedron? For the former, it would be nice to find three planes with dihedral angles $\frac{\pi}{2}$ (for the order-2 rotation about an edge which is always there for a regular polyhedron), $\frac{\pi}{3}$ and $\frac{\pi}{3}$, since the Schläfli symbol for a tetrahedron is $\{3, 3\}$.

Let's do just that! Let P_1, P_2, and P_3, be the planes in $\mathbb{R}^3$ through the origin with normal vectors $n_1 = (1, 1, 0)$, $n_2 = (1, -1, 0)$, and $n_3 = (1, 0, 1)$ respectively. Then the dihedral angles are given by $\theta_{12} = \frac{\pi}{2}$ and $\theta_{13} = \theta_{23} = \frac{\pi}{3}$. Sure enough, if you take the point $(-1, -1, 1)$ (on P_2 and P_3, but not P_1) and reflect across all three planes as much as you can you get the points

$$\{(-1, -1, 1), (-1, 1, -1), (1, -1, -1), (1, 1, 1)\},$$

the vertices of a tetrahedron (see this Desmos graph)!

Lecture 28 - 07/17

Before we go any further, let's lay down some terminology.

Definition. Let $S \subset \text{Isom}(\mathbb{R}^n)$. The *subgroup generated by* S is the set of all isometries formed by compositions of elements of S and their inverses.

Example. If $S \subset \text{Isom}(\mathbb{R}^2)$ is the set of all reflections, then the subgroup generated by S is all of $\text{Isom}(\mathbb{R}^2)$.

Example. If $S = \{\sigma_l, \sigma_m\} \subset \text{Isom}(\mathbb{R}^2)$ where l and m intersect at an angle of $\frac{\pi}{n}$, then the subgroup generated by S has $2n$ elements (justifying this is an exercise).

Definition. Let $G \subset \text{Isom}(\mathbb{R}^n)$ be a subgroup and let $p \in \mathbb{R}^n$. The *orbit* of p under G is the set $G \cdot p = \{g(p) \in \mathbb{R}^n : g \in G\}$.

Example. Let P_1, P_2, and P_3 be the planes through the origin in $\mathbb{R}^3$ with normal vectors $n_1 = (1, 1, 0)$, $n_2 = (1, -1, 0)$, and $n_3 = (1, 0, 1)$ respectively. Let G be the subgroup of $\text{Isom}(\mathbb{R}^3)$ generated by $\left\{\sigma_{P_1}, \sigma_{P_2}, \sigma_{P_3}\right\}$. Then we saw above that if $p = (-1, -1, 1)$, then

$$G \cdot p = \{(-1, -1, 1), (-1, 1, -1), (1, -1, -1), (1, 1, 1)\}.$$

Now, suppose $G \cdot p$ (the orbit of p under some subgroup G of $\text{Isom}(\mathbb{R}^3)$) is a finite set of points. How do we know which subsets of those points form the vertices of one of the faces of a polyhedron? The set of vertices is not enough! However, there is some sense in which the vertices can determine the polyhedra, and that's by taking the *convex hull* of $G \cdot p$.

Definition. Let $\mathcal{P} \subset \mathbb{R}^n$ be a finite set. The *convex hull* of $\mathcal{P}$ is the set

$$\left\{\sum_{p \in \mathcal{P}} \lambda_p p \in \mathbb{R}^n : 0 \leq \lambda_p \leq 1, \sum_{p \in \mathcal{P}} \lambda_p = 1\right\}$$

For some intuition, let's consider the case in $\mathbb{R}^2$. We can imagine stretching an elastic band around all of $\mathcal{P}$ and then letting it snap around the outermost points of $\mathcal{P}$. Everything inside the elastic band is the convex hull of $\mathcal{P}$.

Example. The convex hull of $\{1, -1\} \subset \mathbb{R}$ is the line segment with endpoints 1 and -1.

Example. Let P_1, P_2, and P_3 be the planes in $\mathbb{R}^3$ defined by $x = -y$, $x = y$, and $x = -z$ respectively. Let $G \subset \text{Isom}(\mathbb{R}^3)$ be the subgroup generated by $\left\{\sigma_{P_1}, \sigma_{P_2}, \sigma_{P_3}\right\}$. We saw above that the convex hull of $G \cdot (1, 1, 1)$ is a solid tetrahedron.

Exercise. With G as in the previous example, what is the convex hull of $G \cdot (1, 0, 0)$?

With this language in our back pockets, let's return to constructing regular polyhedra. In order to construct the dodecahedron and icosahedron out of reflections, following the lead from what came before, we need three planes with dihedral angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{5}$. Luckily, such planes exist!

Consider the planes P_1, P_2, and P_3 in $\mathbb{R}^3$ passing through the origin with normal vectors

$$n_1 = (1,0,0), \quad n_2 = (0,1,0), \quad \text{and} \quad n_3 = \left(-2, \sqrt{5}-1, \sqrt{5}+1\right).$$

It can then be checked that the dihedral angles are $\theta_{12} = \frac{\pi}{2}$, $\theta_{13} = \frac{\pi}{3}$, and $\theta_{23} = \frac{\pi}{5}$. There's a hope now of cleverly selecting a point, taking its orbit under the subgroup generated by the three reflections, taking the convex hull, and building an icosahedron or tetrahedron.

Exercise. Let G be the subgroup generated by $\left\{\sigma_{P_1}, \sigma_{P_2}, \sigma(P_3)\right\}$. What is the convex hull of

- $G \cdot (0,1,0)$?
- $G \cdot (-1,-\varphi,0)$?
- $G \cdot \left(0,-\varphi,\frac{1}{\varphi}\right)$?

So, we've been able to construct platonic solids with Schläfli symbol $\{p,q\}$ by finding three planes in $\mathbb{R}^3$ passing through the origin with dihedral angles $\frac{\pi}{2}, \frac{\pi}{p}, \frac{\pi}{q}$. Why can't we build other platonic solids this way?

5.4. Triangles on spheres

Let S^2 be the unit sphere in $\mathbb{R}^3$, that is,

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}.$$

A *line* in S^2 is a great circle, that is, the intersection of S^2 with a plane in $\mathbb{R}^3$ through the origin. The *angle* between two lines is the dihedral angle between the planes defining the lines. Depending on which angle we are measuring, we may want to allow the angle to lie outside the assumed range for dihedral angles.

Example (A spherical triangle). Let $p = (1,0,0)$, $q = (0,1,0)$, $r = (0,0,1)$. There are several triangles with these three points as the vertices. In fact there are eight! Can you draw them all?

The reason that there are many traingles on S^2 with the same three vertices is two fold. First, given a triangle in S^2, we can always take its complement and say that's the triangle! Second, given $p, q \in S^2$, if $p \neq -q$ then there are two possible line segments joining p and q. If $p = -q$, then there are infinitely many!

Returning to our quest to build regular polyhedra, since the dihedral angle between two planes is what we use to define the angle between the corresponding spherical lines, we are interested in finding spherical triangles with internal angles $\frac{\pi}{2}, \frac{\pi}{p}, \frac{\pi}{q}$ for integers $p, q \geq 3$.

To this end, let's investigate what spherical triangles look like. Recall that the surface area of S^2 is 4π.

Definition. A *lune* is a 2-sided polygon on S^2 defined by two great circles. It has two vertices that are antipodal. The *angle of the lune* is the angle between the edges of the lune (and may be $> \pi$).

Since a lune with angle α covers $\frac{\alpha}{2\pi}$ of the sphere, the area of a lune with angle α is 2α.

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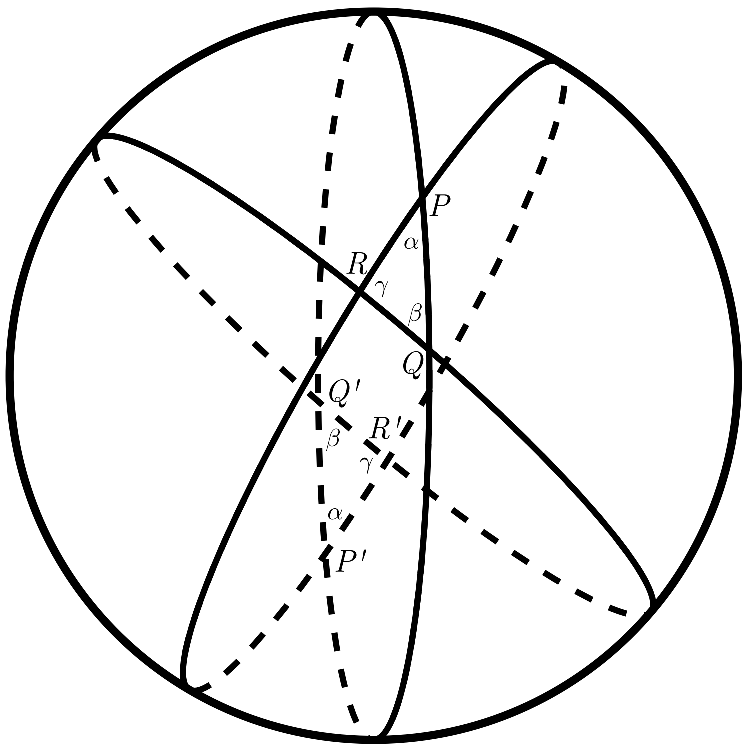
Theorem 24. A spherical triangle with internal angles α, β, γ has area $\alpha + \beta + \gamma - \pi$.

Proof. We begin by first proving the result for triangles where every side length is $\leq \pi$. Consider the triangle T with vertices P, Q, and R, and area A. Let α, β, and γ be the internal angles of T at P, Q, and R respectively. Extend each edge of the triangle to the entire great circle. The three great circles also intersect at $P' = -P$, $Q' = -Q$, and $R' = -R$.

Consider the element $\tau \in \mathrm{Isom}(\mathbb{R}^3)$ given by $\tau(x) = -x$ for all $x \in \mathbb{R}^3$. Then the three great circles divide S^2 up into eight regions, one of which is T and one of which is $\tau(T)$ (the copy of the original triangle at the back of the circle). Since τ is an isometry, the area of T is equal to the area of $\tau(T)$.



The three great circles form two lunes of each angle α, β, and γ. Every point on S^2 is either in T, $\tau(T)$ or exactly one of the six lunes. Furthermore, each of T and $\tau(T)$ is the intersection of exactly one lune of each angle.

So, if we add up the areas of all six lunes, we will account for the surface area of the entire sphere, as well as overcounting the area of T and $\tau(T)$ twice. Therefore

$$4\pi = 4\alpha + 4\beta + 4\gamma - 4A$$

and rearranging gives $A = \alpha + \beta + \gamma - \pi$.

As an exercise, you should complete the proof by using what we've done so far to deal with triangles where one of the side lengths is $> \pi$. $\blacksquare$

The punchline of this theorem is that the sum of the interior angles of a triangle in S^2 is $> \pi$.

So, suppose a spherical triangle has angles $\frac{\pi}{2}, \frac{\pi}{q}, \frac{\pi}{r}$ for integers $q, r > 2$. Then we must have that $\frac{1}{q} + \frac{1}{r} > \frac{1}{2}$. The only possibilities are $(q, r) = (3, 3), (3, 4), (4, 3), (3, 5), (5, 3)$. Do those pairs of numbers look familiar?

Exercise. Let $P \subset S^2$ be a finite set of points, not all on the same great circle. Using the formula for the area of a spherical triangle in terms of its interior angles, prove that the sum of the angle defects of the convex hull of P is 4π.

Lecture 30 - 07/22

5.5. Coxeter diagrams

You may have noticed that in the constructions of the regular polyhedra above by using reflections, the specific planes don't actually matter. What matters is the dihedral angles between them. Furthermore, when selecting a point whose orbit to consider, the specific point is not important, but it is important which planes the point lies on.

We can encapsulate the important information in something called a *Coxeter diagram*.

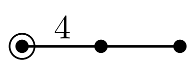
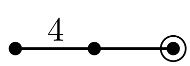
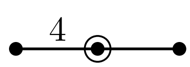
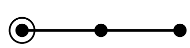
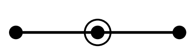
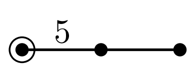
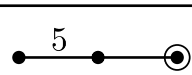
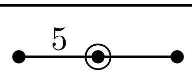
A Coxeter diagram is a graph (so a collection of vertices and edges) where the vertices correspond to reflections across $(n-1)$-dimensional planes in $\mathbb{R}^n$. Each edge may be labelled by an integer ≥ 4. Here are the conventions for interpreting a Coxeter diagram:

- If there is no edge between two vertices, then the corresponding reflections are across planes that have a dihedral angle of $\frac{\pi}{2}$.
- If there is an unlabelled edge, the dihedral angle between the corresponding planes is $\frac{\pi}{3}$.
- If an edge is labelled by $n \geq 4$, then the dihedral angle between the planes is $\frac{\pi}{n}$.

The labelled graph is a *Coxeter diagram* and denotes the subgroup of $\text{Isom}(\mathbb{R}^n)$ generated by the reflections.

If a vertex has a ring around it, the Coxeter diagram denotes the convex hull of the orbit of a point that is on all planes not ringed.

The following table gathers up what we've done so far.

Coxeter diagram	**Polyhedron**
	Cube
	Octahedron
	?????
	Tetrahedron
	?????
	Dodecahedron
	Icosahedron
	?????

Exercise. Fill in the missing cells in the table above, with names of polyhedra from this wonderful website.

Exercise. Consider the coxeter diagram from the first row of the table above, corresponding to the cube. Now ring the middle vertex as well. This denotes some polyhedron, realised as the convex hull of $G \cdot p$ where p is some point on the unringed plane, and G is the group denoted by the unringed Coxeter diagram.

- Where should p be in order for the faces of the polyhedron to all be regular polygons?
- What are the faces in this case?

5.6. Higher-dimensional Polytopes

Definition. A *polytope* is the general term for an element of the sequence

point, line segment, polygon, polyhedron, …

Lecture 31 - 07/24

The n-cube

- A 0-cube is a point.
- A 1-cube is a line segment, which can be realised with vertices $\{-1, 1\} \subset \mathbb{R}$
- A 2-cube is a square, which can be realised with vertices $\{(\pm 1, \pm 1)\} \subset \mathbb{R}^2$.
- A 3-cube is a cube, which can be realised with vertices $\{(\pm 1, \pm 1, \pm 1)\} \subset \mathbb{R}^3$.
- A 4-cube is a *tesseract* or *hypercube*, which can be realised with vertices $\{(\pm 1, \pm 1, \pm 1, \pm 1)\} \subset \mathbb{R}^4$.

In a hypercube, the 3-dimensional faces, called *cells* are all cubes, and there are eight of them. There are 24 faces (which are squares), 32 edges, and 16 vertices.

In general, an n-polytope has k-faces for each $k \in \{0, 1, ..., n-1\}$. 0-faces are vertices, 1-faces are edges, 2-faces are faces, and 3-faces are cells.

Exercise. How many k-faces are there in an n-cube?

The n cube is an example of a regular n-polytope.

Definition. A polytope is *regular* if for any two k-faces (both with the same k value), there is a symmetry of the polytope that maps one to the other.

There are lots of consequences of being regular. In particular, each k-face is a regular k-polytope, and all the k-faces are the *same* regular k-polytope. The same number of $(n-1)$-faces meet at every vertex. The same number of cells meet at every edge. Pretty much anything you can measure is the same everywhere!

In a hypercube, three cubes meet at every edge, as do three faces, and four cubes meet at every vertex.

The cross polytope

This is a high-dimensional version of an octahedron, and is the dual of the n-cube (whatever dual means in higher dimensions, see if you can come up with an appropriate definition).

- A 1-cross polytope is a line segment, which can be realised with vertices $\{\pm 1\} \subset \mathbb{R}$.
- A 2-cross polytope is a diamond, which can be realised with vertices $\{\pm e_1, \pm e_2\} \subset \mathbb{R}^2$.

- A 3-cross polytope is an octahedron, which can be realised with vertices $\{\pm e_1, \pm e_2, \pm e_3\} \subset \mathbb{R}^3$.

The 4-cross polytope (which also goes by the names *16-cell*, *hexadecachoron*, and *4-orthoplex*) can be realised with vertices $\{\pm e_1, ..., \pm e_4\} \subset \mathbb{R}^4$. It has 16 tetrahedral cells, 32 triangular faces, 24 edges and 8 vertices.

Exercise. How many cells meet at each edge? How many k-faces are there in an n-cross polytope?

The simplex

The n-simplex can be realised as the convex hull of the standard basis vectors $\left\{e_1, e_2, e_3, ..., e_{\{n+1\}}\right\}$ in $\mathbb{R}^{n+1}$. So, a 0-simplex is a point, a 1-simplex a line segment, a 2-simplex a triangle, and a 3-simplex a tetrahedron.

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The 4-simplex (or as it's sometimes known, the 5-cell) is the convex hull of $\{e_1, e_2, e_3, e_4, e_5\}$ in $\mathbb{R}^5$. The graph of vertices and edges is the complete graph on 5 vertices. Every subset of 4 vertices forms a cell, each of which is a tetrahedron. Every subset of 3 vertices forms a face, which is a triangle. Every subset of 2 vertices forms an edge.

With this in mind, the 5-cell has 5 tetrahedral cells (hence the name), 10 triangular faces, 10 edges, and 5 vertices.

Exercise. How many k-faces are in an n-simplex?

It has not been proved yet (although you have all the tools to do so), that the n-cube, n-cross polytope, and n-simplex are all regular polytopes for all n.

Exercise. Let X be either the n-cube, n-cross polytope, or the $(n-1)$-simplex. Let v and w be two vertices in X. Prove that there exists an $(n-1)$-dimensional plane P in $\mathbb{R}^n$ so that the reflection σ_P satisfies $\sigma_{P(v)} = w$.

Note: This exercise does not prove that these polytopes are regular, but it does prove that for any two 0*-faces, there is a symmetry of* X *taking one to the other.*

We have the following perhaps surprising fact, which we shall unfortunately not prove in this course.

Fact 25. For $n \geq 5$, the only regular n-polytopes are the n-cube, the n-simplex, and the n-cross polytope.

5.7. The return of Schläfli

There is only one regular 1-polytope, which is a line segment. Every regular n-gon is a regular 2-polytope, and these are all the regular 2-polytopes. There are five regular 3-polytopes, namely the platonic solids. For $n \geq 5$, there are exactly three regular n-polytopes, the n-cube, the n-simplex, and the n-cross polytope. All that remains is the $n = 4$ case, which we will approach by bringing back our old friend, the Schläfli symbol.

Definition. The *Schläfli symbol* of a regular 4-polytope is $\{p, q, r\}$, where $\{p, q\}$ is the type of the cell, and r cells meet at each edge.

So, for example, $\{4, 3, 3\}$ denotes the hypercube, $\{3, 3, 3\}$ denotes the 4-simplex, and $\{3, 3, 4\}$ denotes the 16-cell (or the 4-cross polytope).

Now, just like we needed the angle defect of a regular polyhedron to be positive, we have a similar restriction on the dihedral angles that meet at the edge of a regular 4-polytope.

Fact 26 (Schläfli's criterion). The sum of the dihedral angles of cells meeting at an edge of a regular 4-polytope must sum up to $< 2\pi$.

We won't prove this here, but the idea is that if the sum of the dihedral angles that meet at an edge is equal to 2π, then the polytope will not curve back on itself to form some bounded polytope. This is analogous to how if the sum of the interior angles of polygons meeting at a vertex is 2π, then those polygons tile the plane, and don't form a polyhedron.

In order to find all the possible regular 4-polytopes, let's first see what restrictions Schläfli's criterion puts on the possible Schläfli symbols.

Suppose a regular 4-polytope has Schläfli symbol $\{p, q, r\}$. Since the polytope is regular, each cell is regular, and thus $\{p, q\} = \{3, 3\}, \{3, 4\}, \{4, 3\}, \{3, 5\},$ or $\{5, 3\}$. To put restrictions on r, we need to know the dihedral angles of the faces meeting at an edge of each of the platonic solids. Here they are

Platonic solid	**Schläfli symbol**	**Dihedral angle**
Tetrahedron	$\{3, 3\}$	$\arccos\left(\frac{1}{3}\right) \approx 70.53^\circ$
Cube	$\{4, 3\}$	$\frac{\pi}{2}$
Octahedron	$\{3, 4\}$	$\arccos\left(-\frac{1}{3}\right) \approx 109.47^\circ$
Dodecahedron	$\{5, 3\}$	$\arccos\left(-\frac{1}{\sqrt{5}}\right) \approx 116.57^\circ$
Icosahedron	$\{3, 5\}$	$\arccos\left(-\frac{\sqrt{5}}{3}\right) \approx 138.19^\circ$

Exercise. Verify that these are indeed the dihedral angles of edges of the platonic solids.

If there are r cells meeting at each edge, and each dihedral angle of an edge of a cell is d, then Schläfli's criterion tells us $rd < 2\pi$. Since we must have $r > 2$ (can you see why?), we have the following result.

Lemma 27. The possible Schläfli symbols for regular 4-polytopes are

$$\{3, 3, 3\}, \{3, 3, 4\}, \{3, 3, 5\}, \{4, 3, 3\}, \{3, 4, 3\}, \{5, 3, 3\}.$$

The symbols $\{3, 3, 3\}, \{3, 3, 4\}$, and $\{4, 3, 3\}$ are accounted for by the 5-cell, the 16-cell and the hypercube respectively. That leaves us with the possible existence of three other regular 4-polytopes, with Schläfli symbols $\{3, 3, 5\}$, $\{3, 4, 3\}$, and $\{5, 3, 3\}$.

These will exist, but to see them, we must welcome back another old friend.

Lecture 33 - 07/29

5.8. The return of Coxeter

It will turn out that all six Schläfli symbols are realised by a regular 4-polytope. Instead of simply writing out the coordinates of the vertices (which will turn out to be quite cumbersome since one of the polytopes has 600 vertices), we will construct each one out of reflections. As Coxeter intended.

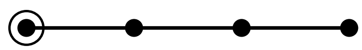
Let's start by constructing the 5-cell $\{3, 3, 3\}$. Consider the planes $P_1, P_2, P_3, P_4 \subset \mathbb{R}^5$ defined by

$$P_i = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_i = x_{i+1}\}.$$

Then it can be checked (and is an exercise for you to do so) that these four planes satisfy that the dihedral angle between P_i and P_{i+1} is $\frac{\pi}{3}$, and all other dihedral angles are $\frac{\pi}{2}$.

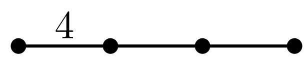
Now, if G is the subgroup of $\text{Isom}(\mathbb{R}^5)$ generated by the reflections $\{\sigma_{P_1}, \sigma_{P_2}, \sigma_{P_3}, \sigma_{P_4}\}$, then we have that $G \cdot e_1 = \{e_1, e_2, e_3, e_4, e_5\}$ (where e_i is the ith standard basis vector in $\mathbb{R}^5$).

Putting all of this in a neat little package, and noting that e_1 is in P_2, P_3, P_4 and not P_1, we see that the 5-cell is given by the following Coxeter diagram:

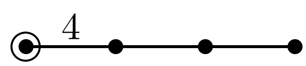


Exercise.

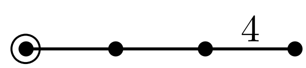
1. Find planes P_1, P_2, P_3, P_4 in $\mathbb{R}^4$ through the origin that give the Coxeter diagram



2. Show that the 4-cube (with Schläfli symbol $\{4, 3, 3\}$) is defined by the Coxeter diagram



3. Show that the 4-cross polytope (with Schläfli symbol $\{3, 3, 4\}$) is defined by the Coxeter diagram



In every example we have of a regular polytope (2-, 3-, and 4-dimensional), notice the similarities between the Schläfli symbol and the Coxeter diagram. In every case we have computed so far (or that has been left as an exercise), the Coxeter diagram corresponding to the polytope with Schläfli symbol $\{p_1, p_2, p_3, \dots, p_k\}$ is a graph that is a line, with the left-most vertex ringed, and labels (from left to right) $p_1, p_2, \dots, p_k$.

With this in mind, let's try constructing the regular 4-polytopes with Schläfli symbols $\{3, 4, 3\}$, $\{3, 3, 5\}$, and $\{5, 3, 3\}$.

The 24-cell

Let's attempt to build the regular 4-polytope with Schläfli symbol $\{3, 4, 3\}$. Spoiler: It exists and is called the 24-cell because, well, it has 24 cells. Creative, huh?

Consider the following four planes in $\mathbb{R}^4$:

$$P_1 = \{(w, x, y, z) \in \mathbb{R}^4 : x = y\}$$
$$P_2 = \{(w, x, y, z) \in \mathbb{R}^4 : y = z\}$$
$$P_3 = \{(w, x, y, z) \in \mathbb{R}^4 : z = 0\}$$
$$P_4 = \{(w, x, y, z) \in \mathbb{R}^4 : w - x - y - z = 0\}.$$

These planes have normal vectors

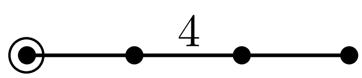
$$
\begin{aligned}
n_1 &= (0, 1, -1, 0) \\
n_2 &= (0, 0, 1, -1) \\
n_3 &= (0, 0, 0, 1) \\
n_4 &= (1, -1, -1, -1).
\end{aligned}
$$

Computing the dihedral angles θ_{ij} between P_i and P_j we have

$$\theta_{12} = \frac{\pi}{3}, \quad \theta_{23} = \frac{\pi}{4}, \quad \theta_{34} = \frac{\pi}{3}, \quad \text{and} \quad \theta_{ij} = \frac{\pi}{2}$$

for all other cases. This is a promising start! Now let's choose a point p on P_2, P_3, and P_4, but not on P_1 (this will correspond to putting a ring around the vertex in the Coxeter diagram corresponding to P_1). A perfectly reasonable choice of point is $p = (1, 1, 0, 0)$.

Now, if G is the subgroup of $\mathrm{Isom}(\mathbb{R}^4)$ generated by the reflections $\left\{\sigma_{P_1}, \sigma_{P_2}, \sigma_{P_3}, \sigma_{P_4}\right\}$, then the convex hull of $G \cdot p$ is the 24-cell! That is, it's a regular 4-polytope with Schläfli symbol $\{3, 4, 3\}$ and Coxeter diagram as follows:



We can painstakingly compute the vertices of the 24-cell by computing the orbit of p under G. Indeed, we get

$$G \cdot p = \left\{\varepsilon e_i + \mu e_j \in \mathbb{R}^4 : \varepsilon, \mu \in \{1, -1\} \text{ and } 1 \leq i < j \leq 4\right\}.$$

In more simple terms, the 24 vertices are all points where two of the coordinates are 0 and the other two coordinates are 1 or -1.

Exercise. Each cell of the 24-cell is an octahedron. Find all 24 six-element subsets of the 24 vertices of the 24-cell that form the vertices of the 24 octahedra.

Let's investigate the 24-cell a little more. Since the Schläfli symbol is a palindrome, the 24-cell is its own dual! This checks out with the fact that there are the same number of vertices and cells. Being self dual also implies that there are the same number of faces and edges.

The Schläfli symbol tells us that every cell is an octahedron (which has 12 edges) and three cells meet at every edge. Therefore, the number of edges E is given by

$$E = \frac{12 \cdot 24}{3} = 96.$$

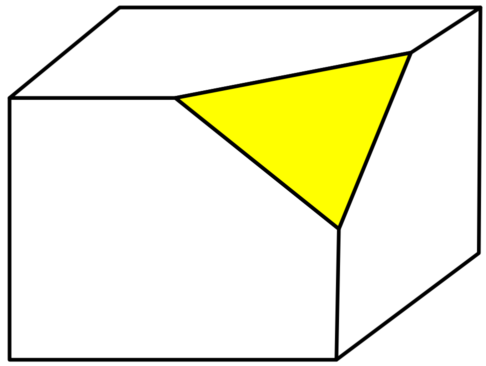
And we also conclude that there are 96 (triangular) faces.

Exercise. Prove that the dual of $\{p, q, r\}$ is $\{r, p, q\}$.

Lecture 34 - 07/31

There is another feature of a regular polytope which can be read off from the Schläfli symbol. We need to introduce it here before investigating the final two regular polytopes.

Consider the cube, which has Schläfli symbol $\{4, 3\}$. The 3 tells us that three faces meet at each vertex. But there is another way to interpret the 3 here. Imagine you took a knife and sliced off one of the vertices from the other eight. The shape of the slice on the cube would be a triangle (which has 3 sides).



In this case we say the *vertex figure* of the cube is a triangle.

Here is a more formal definition.

Definition. The *vertex figure* of a vertex of a regular polytope is the convex hull of the midpoints of all the adjacent edges.

Exercise. What is the vertex figure of a dodecahedron? What about the 4-simplex?

Proposition 28. The vertex figure of a regular n-polytope is a regular $(n-1)$-polytope.

Proof. This is an exercise. ∎

Consider the hypercube. Every vertex is adjacent to exactly four other vertices (each obtained by reversing the sign of exactly one of the coordinates). Therefore, three edges meet at each vertex. We can then conclude that the vertex figure of the hypercube is the convex hull of four points, and therefore must be a tetrahedron.

Knowing the vertex figure allows us to conclude many things. A tetrahedron has four vertices, six edges, and four faces. This implies that at each vertex of the hypercube, four edges, six faces, and four cells (respectively) meet.

Another thing worth noting in this example is that the Schläfli symbol of the vertex figure of $\{4,3,3\}$ is $\{3,3\}$.

Proposition 29. The vertex figure of $\{p,q,r\}$ is $\{q,r\}$.

Proof. This is an exercise. ∎

So from the Schläfli symbol for a regular 4-polytope, we can read of the platonic solid that appears as each cell, as well as the platonic solid that appears as the vertex figure. We can get a lot of mileage out of this!

Let's return for a moment to the hypercube, and let's say we only know its Schläfli symbol was $\{4,3,3\}$ and that it has 16 vertices.

Each cell is a cube, which has 8 vertices. By knowing the vertex figure is a tetrahedron, which has four faces, we know four cells meet at each vertex. Therefore the number of cells C satisfies

$$16 = \frac{8C}{4} = 2C$$

and so we have that $C = 8$. Since three cells meet at each edge, and each cell has 12 edges (since they are cubes), we have that the number of edges E satisfies

$$E = \frac{8 \cdot 12}{3} = 32.$$

Finally, since two cells meet at each face, and each cell has six faces, the number of face F is given by

$$F = \frac{8 \cdot 6}{2} = 24.$$

By only knowing the number of vertices and the Schläfli symbol, we can compute C, E, and F for the hypercube.

The 120-cell and the 600-cell

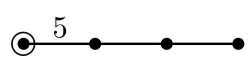
The main difficulty in showing that the polytopes with Schläfli symbols $\{3, 3, 5\}$ and $\{5, 3, 3\}$ exist, is the existence of planes with the appropriate dihedral angles. To do this, let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, and consider the vectors

$$n_1 = (0, 1, 0, 0), \quad n_2 = (0, \varphi, \varphi^{-1}, 1), \quad n_3 = (0, 0, 0, 1), \quad \text{and} \quad n_4 = (\varphi^{-1}, 0, -\varphi, 1).$$

Exercise. Show that the angles θ_{ij} between n_i and n_j are given by

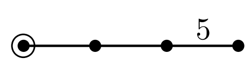
$$\theta_{12} = \frac{\pi}{5}, \quad \theta_{23} = \theta_{34} = \frac{\pi}{3}, \quad \text{and} \quad \theta_{13} = \theta_{24} = \theta_{14} = \frac{\pi}{2}.$$

Let P_i be the plane in $\mathbb{R}^4$ with normal vector n_i, and let G be the subgroup of $\mathrm{Isom}(\mathbb{R}^4)$ generated by the reflections $\left\{\sigma_{P_1}, \sigma_{P_2}, \sigma_{P_3}, \sigma_{P_4}\right\}$. Then for any point p on P_2, P_3, P_4 and not on P_1 (for example $p = (\varphi^{-2}, 1, -\varphi^2, 0)$) we have that the convex hull of $G \cdot p$ has Coxeter symbol



and Schläfli symbol $\{5, 3, 3\}$. This polytope is called the **120-cell**.

If instead we choose a point p on P_1, P_2, P_3 but not on P_4 (for example, $p = (1, 0, 0, 0)$) we have that the convex hull of $G \cdot p$ has Coxeter symbol



and Schläfli symbol $\{3, 3, 5\}$. This polytope, the dual of the 120-cell, is called the **600-cell**.

The 120-cell and 600-cell have 120 dodecahedral and 600 tetrahedral cells respectively, and vertex figures that are tetrahedra and icosahedra respectively.

Exercise. Taking for granted that the 120-cell and the 600-cell have 120 and 600 cells respectively, calculate the number of verices V, edges E, and faces F of both polytopes.

Of course, if you really wanted to, you can now generate coordinates for the vertices of both these polytopes (with the help of a computer of course), and count the number of vertices of each. Since they are duals of each other, you should find that the 600-cell has 120 vertices, and the 120-cell has 600 vertices (which should agree with your answers from the previous exercise).

A. Euclid's definitions, postulates, and common notions

These are taken from the wonderful website http://aleph0.clarku.edu/~djoyce/elements/bookI/bookI.html

Definitions

1. A point is that which has no part.
2. A line is breadthless length.
3. The ends of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The edges of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
16. And the point is called the center of the circle.
17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.
19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.
20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.
22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles equal one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.