Problem of the Month Lesson
Binomial coefficients and the binomial theorem

Let's start by thinking about the following two problems:

Problem 1: *How many different seven-digit numbers are there that use exactly three 5s and four 2s?*

Problem 2: *After expanding and simplifying the expression $(a+b)^7$ we get*

$$(a+b)^7 = Sb^7 + Tab^6 + Ua^2b^5 + Va^3b^4 + Wa^4b^3 + Xa^5b^2 + Ya^6b + Za^7$$

for some positive integers S, T, U, V, W, X, Y, Z. Compute V.

At first glance, these problems appear quite different. If we allow ourselves to delve a little deeper, we will find that they are tied together by the use of *binomial coefficients*. Before continuing, give yourself a chance to think about, and perhaps even solve, these problems.

Solving Problem 1.

Here is a way to approach this problem. You are trying to place three 5s in seven available spots. Once you have placed the three 5s, then there is no choice as to where to put the four 2s. So the question is now, how many ways are there to place three 5s in seven spots?

Let's place the 5s one at a time. For the first 5, there are seven possibilities of where it goes. Once you have placed that 5, there are six available places for the second 5. Finally, there are five available places for the third 5. For example, if the first, second, and third 5s go in the second, seventh, and fourth spots respectively, the resulting seven-digit number is 2525225.

It is tempting to now claim that the answer is $7 \cdot 6 \cdot 5$. However, this count is too high! For example, placing the three 5s in the second, seventh, and fourth position (in that order) yields the seven-digit number 2525225. This is the same number obtained by placing the three 5s in the second, then first, then fourth position, or the fourth, then first, then second position.

In fact, any ordering of the three positions of the 5s gives the same seven-digit number! So, in how many different orders can you place the three 5s in the same positions? Well, the first 5 has three possible places it can go. After the first is placed, the second only has two possible positions. Finally, the last 5 can only go in the one remaining position. Therefore, there are $3 \cdot 2 \cdot 1 = 6$ different ways to obtain each seven-digit number.

So, our initial count of $7 \cdot 6 \cdot 5$ is too high, and to get the actual count, we must divide this by $3 \cdot 2 \cdot 1$. Our final answer is

$$\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

If you have twenty minutes to kill later this evening, see if you can list all 35 seven-digit numbers!

Factorials and binomial coefficients.

In our solution to Problem 1, we found the number of ways to choose 3 things from a set of 7 things is 35. This kind of count comes up often enough in a variety of mathematical contexts, that we give it a name and some notation.

Let n and k be integers satisfying $0 \leq k \leq n$. The *binomial coefficient* $\binom{n}{k}$ is the number of ways to choose k objects from a set of n distinct objects.

In the solution above, we argued that the number of seven-digit numbers consisting of exactly three 5s and four 2s is the binomial coefficient $\binom{7}{3}$. Even better, we calculated its value and showed that

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

The expression $3 \cdot 2 \cdot 1$ in the denominator is a count of how many different ways you can order three distinct objects. Expressions like this also occur frequently enough to warrant special notation.

For a positive integer n, the integer n *factorial*, denoted $n!$ is defined to be

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

The integer $n!$ counts the number of ways to arrange n distinct items in some order. For example, the number of four-digit numbers that use each of the digits $3, 5, 7, 9$ exactly once is $4! = 24$. In order for things to not get out of hand, we define $0! = 1$ (you should think about why we choose to do this!).

We can use factorials to express the binomial coefficient $\binom{n}{k}$. Generalising the argument made in the solution to Problem 1, we have

$$\begin{aligned}
\binom{n}{k} &= \frac{n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1)}{k \cdot (k-1) \cdots 2 \cdot 1} \\
&= \frac{\big(n \cdot (n-1) \cdots (n-k+2) \cdot (n-k+1)\big)\big((n-k) \cdot (n-k-1) \cdots 2 \cdot 1\big)}{k!\big((n-k) \cdot (n-k-1) \cdots 2 \cdot 1\big)} \\
&= \frac{n!}{k! \cdot (n-k)!}.
\end{aligned}$$

This is a fantastically useful formula, and is worth playing around with. As a first exercise, see if you can prove that $\binom{n}{k} = \binom{n}{n-k}$.

Solving Problem 2.

An approach to this problem is to simply roll up our sleeves, and just expand out the expression $(a+b)^7$ one bracket at a time. This approach will certainly work, but it's a lot of effort with many opportunities for simple errors to occur. Let's see if we can do something a little less like a calculator, and a little more like a human.

Let's think carefully about a simpler, much more familiar expansion:

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Where does the coefficient 2 come from in front of the ab? To really understand this completely, let's show some steps in this expansion.

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= aa + ab + ba + bb \\
&= a^2 + 2ab + b^2.
\end{aligned}$$

When we expand out the parentheses, we choose exactly one of either a or b from each set of parentheses. The second line in the expansion shows which of a or b is being chosen from each set of parentheses. The term ba, for example, indicates that b is chosen from the first set of parentheses, and the a from the second.

The reason why the coefficient of ab is 2, but the coefficient of a^2 is 1, is now revealed. It is because there are two ways to choose one a and one b from the two sets of parentheses, but only one way to choose two as.

Great! Let's apply this observation to Problem 2. The integer V is the coefficient in front of a^3b^4. So V counts the number of ways to choose three as (and therefore four bs) from the seven parentheses. We've done this before! This is precisly the binomial coefficient $\binom{7}{3}$, so $V = 35$. Neat!

The binomial theorem

We can go further with our solution to Problem 2 and compute all of the coefficients this way! The integer S counts the number of ways to choose a zero times from the seven parentheses (which of course is only 1). The integer T counts the number of ways to choose one a from the seven parentheses, and so on. Therefore,

$$\begin{aligned}(a+b)^7 &= \binom{7}{0}b^7 + \binom{7}{1}ab^6 + \binom{7}{2}a^2b^5 + \binom{7}{3}a^3b^4 + \binom{7}{4}a^4b^3 + \binom{7}{5}a^5b^2 + \binom{7}{6}a^6b + \binom{7}{7}a^7 \\ &= b^7 + 7ab^6 + 21a^2b^5 + 35a^3b^4 + 35a^4b^3 + 21a^5b^2 + 7a^6b + a^7.\end{aligned}$$

Of course, there is nothing special about the exponent 7 here, and the more general statement is what's known as the ***binomial theorem***:

Let $n \geq 0$ be an integer. Then

$$(a+b)^n = \binom{n}{0}b^n + \binom{n}{1}ab^{n-1} + \binom{n}{2}a^2b^{n-2} + \cdots + \binom{n}{n-1}a^{n-1}b + \binom{n}{n}a^n.$$

By cleverly choosing a and b you can quickly prove some very interesting identities. For example, if $a = b = 1$ we get that for all non-negative integers n,

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

What happens if you let $a = 1$ and $b = -1$?

Pascal's triangle

Now for something completely different. Well, different at first, anyway. Let's make a triangle!

We will start at the top, with the number 1. The second row will have two numbers, the third row will have three numbers, and so on. To fill in the numbers in the triangle, we fill in the first and last entry of each row with a 1, and insist that all other entries are the sum of the two above it. This triangle is called Pascal's triangle, and the first few rows look like this:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

The third row is $1, 2, 1$, which happen to be the coeffients of the expansion of $(a+b)^2$. The eighth row is $1, 7, 21, 35, 35, 21, 7, 1$, which are the coefficients in the expansion of $(a+b)^7$. Coincidence? No!

It turns out that the entries in Pascal's triangle are exactly the binomial coefficients. Here is the same triangle:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}
$$

Pascal's triangle is a great way to quickly compute binomial coefficients with small integers. One important step in proving that Pascal's triangle is indeed a triangle full of binomial coefficients, is proving the following statement known as Pascal's identity:

For all integers n and k satisfying $0 \leq k < n$,

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

See if you can prove Pascal's identity in two different ways. One way is by using the formula for the binomial coefficient.

Another way is by thinking about what the binomial coefficients are counting. See if you can convince yourself (or even better, someone else!) that the number of ways to choose $k+1$ objects from $n+1$ distinct objects is equal to the sum of two things: the number of ways to choose k objects from n distinct objects, and the number of ways to choose $k+1$ objects from n distinct objects.