PMATH 340 - Elementary Number Theory
Course Notes, University of Waterloo

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Contents

These notes are an overview of what was covered in each lecture of the course. They will be updated as I go, and are definitely not free of typos and mistakes. If you find any, please let me know about it and I'll fix them as soon as possible.

1 Divisibility

Divisibility is easy in the real numbers or the rational numbers. Things are a little more interesting when we deal with the integers $\mathbb{Z}$.

Definition. Let $a, b \in \mathbb{Z}$. We say a **divides** b, and write $a \mid b$, if there exists an integer k so that $ak = b$.

If $a \mid b$, we can also say a is a factor of b or b is a multiple of a.

Example.

- $3 \mid 6$ since $3 \cdot 2 = 6$ (and 2 is an integer!).

- $3 \mid -3$ since $3 \cdot (-1) = -3$.

- $27 \mid 0$ since $27 \cdot 0 = 0$.

- $1 \mid a$ for all $a \in \mathbb{Z}$ since $1 \cdot a = a$.

Let's prove something about divisibility.

Theorem 1. *If $d \mid a$ and $a \neq 0$, then $|d| \leq |a|$.*

Proof. Since $d \mid a$, $dk = a$ for some $k \in \mathbb{Z}$. Therefore $|a| = |dk| = |d| \, |k|$. Since $a \neq 0$, $k \neq 0$ so $|k| \geq 1$. Multiplying both sides of this inequality by the positive number $|d|$ gives $|d| \, |k| \geq |d|$. Therefore $|a| \geq |d|$, completing the proof. ∎

Notice that the theorem is false if $a = 0$ (for example if $k = 3$ and $a = 0$), so we really do need the assumption $a \neq 0$. Here are some facts about divisibility that we will use, but will be left as an exercise.

Exercise. Prove the following statements.

- If $c \mid a$ and $c \mid b$, then $c \mid av + bu$ for all $u, v \in \mathbb{Z}$.

- If $a \mid b$ and $b \mid c$, then $a \mid c$.

- If $a \mid b$, then $a^2 \mid b^2$.

Theorem 2. *Let $a, b \in \mathbb{Z}$. We have $a \mid b$ and $b \mid a$ if and only if $a = \pm b$.*

Proof. If $a = \pm b$, then we have $a(\pm 1) = b$ and $b(\pm 1) = a$, so $a \mid b$ and $b \mid a$. Conversely, suppose $ak = b$ and $bl = a$ for some $k, l \in \mathbb{Z}$. Then $akl = a$. If $a = 0$, then $b = ak = 0$, so $a = \pm b$. Therefore we may assume $a \neq 0$, so $kl = 1$. Then $k = l = \pm 1$ so $a = \pm b$. ∎

We could have also proved this theorem by dealing with the case $a = b = 0$ first, and then using Theorem 1 to get $|a| \leq |b|$ and $|b| \leq |a|$. We could then conclude $|a| = |b|$ so $a = \pm b$.

Exercise. Prove or disprove: If $a \mid b$ and $c \mid d$, then $a + c \mid b + d$.

Now, we know not every integer divides every other integer (for example $4 \nmid 5$). However, we can still say something in this case. Let's draw out the number line and emphasise all the multiples of 4.
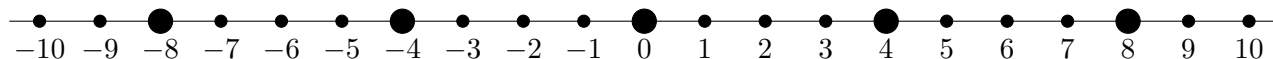
Notice that although there are a bunch of numbers which aren't multiples of 4, then all either land on a multiple of 4, or either 1, 2, or 3, places to the right of a multiple of 4 (how far to the right is what we would call the remainder when divided by 4). So for example, $4 \nmid 5$ but we do have that 5 is one more than a multiple of 4. In fact, looking at this line, we can see every integer a can be written as $a = q \cdot 4 + r$ where q is some integer and $0 \le r < 4$. Let's prove this is always the case!

Theorem 3 (The Division Algorithm). *If $a, b \in \mathbb{Z}$ with $b > 0$, then there is a unique pair of integers q, r such that $a = qb + r$ and $0 \le r < b$.*

If $a = qb + r$ as per the division algorithm, we call the integer r the **remainder** of a when divided by b.

Before we embark on the proof, we have to first talk about the well-ordering principle. In this course, the natural numbers are

$$\mathbb{N} = \{0, 1, 2, \ldots\}.$$

The well-ordering principle: Every non-empty subset of $\mathbb{N}$ has a least element.

This is an axiom of the natural numbers (that is, something that is handed to us as a defining feature of $\mathbb{N}$, not something we can prove). Not every set has this property! For example, $\mathbb{Z}$ doesn't have a least element (and $\mathbb{Z}$ is a non-empty subset of, well, $\mathbb{Z}$), so $\mathbb{Z}$ does not satisfy the well-ordering principle.

Exercise. Show that $\mathbb{Q}$ does not satisfy the well-ordering principle. *Hint: Consider the set $\{x \in \mathbb{Q} \mid x > 0\}$.*

Now let's prove the Division Algorithm, and we will rely on the well-ordering principle to do so.

Proof. Let's first show such a q and r exist. Let $S = \{a - nb \mid n \in \mathbb{Z}\}$. If $n = -|a|$ then $a + |a| b \ge 0$ so $S \cap \mathbb{N}$ is a non-empty subset of $\mathbb{N}$. Let r be the least element of $S \cap \mathbb{N}$, so $r = a - qb \ge 0$ for some $q \in \mathbb{Z}$. Rearranging gives $a = qb + r$ with $r \ge 0$. We want to show $r < b$. Towards that goal, suppose $r \ge b$. Then $r - b \ge 0$ so $a - (q + 1)b = r - b \ge 0$, which contradicts r being the least element in $S \cap \mathbb{N}$. Therefore $0 \le r < b$.

For uniqueness, suppose $a = qb + r = qb' + r'$ with $0 \le r, r' < b$. Then $r - r' = (q' - q)b$. Since $0 \le r, r' < b$, we have $|r - r'| < |b|$. Therefore $|r - r'| = |q - q'| \, |b|$ implies $|q - q'| < 1$. Since $q, q' \in \mathbb{Z}$ we must have $q = q'$ and thus $r = r'$. ∎

The proof was hard, but the theorem can actually do quite a bit of heavy lifting for us!

Example. Suppose $3 \nmid a$. Then the remainder of a^2 when divided by 3 is 1. Let's investigate this claim. If $3 \nmid a$, then $a = 3q + r$ where $r = 1$ or 2 and $q \in \mathbb{Z}$. If $r = 1$, then

$$a^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1.$$

if $r = 2$ then
$$a^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1.$$
In both cases, the remainder is 1, which justifies the initial claim!

Exercise.

- What are all the remainders of squares of odd numbers when divided by 8?

- Prove that an odd number times an odd number is always odd.

Lecture 3 - 10/01

1.1   The Greatest Common Divisor

An extremely useful concept to pay attention to turns out to be the greatest common divisor of two integers. Intuitively, this is how much the two numbers share. You already have experience dealing with this concept when you simplify fractions. For example, you usually wouldn't write $\frac{10}{4}$, but instead you would write $\frac{5}{2}$. In the former case, 2 divides both 10 and 4, and it's the biggest thing that divides both, so you divide the numerator and denominator by 2 to obtain $\frac{5}{2}$. Even better, 5 and 2 don't share any divisors (other than ± 1), so you cannot simplify this fraction any further.

Definition. Let $a, b \in \mathbb{Z}$. A **common divisor** of a and b is an integer c so that $c \mid a$ and $c \mid b$.

Definition. Let $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$. Define the **greatest common divisor** of a and b, denoted $\gcd(a, b)$, to be the integer d such that

- d is a common divisor of a and b, and

- if c is a common divisor of a and b, then $c \leq d$.

We omit the case $a = b = 0$, and leave $\gcd(0, 0)$ to be undefined. After all, every integer divides 0, so there is no candidate for the greatest common divisor!

Example.

- $\gcd(2, 4) = 2$.

- $\gcd(5, 9) = 1$.

- $\gcd(-5, 9) = 1$.

- $\gcd(0, a) = |a|$ for all $a \in \mathbb{Z} \setminus \{0\}$.

- $\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b)$ for all $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$.

Notice that if $c \mid a$ and $c \mid b$, then $-c \mid a$ and $-c \mid b$. Therefore we know $\gcd(a, b) \geq 0$. In the definition of the greatest common divisor we called it "the" greatest common divisor instead of "a" greatest common divisor, suggesting it is unique. Indeed it is!

Theorem 4. *When defined, the greatest common divisor of two integers is unique.*

Proof. Let $a, b \in \mathbb{Z}$, $a \neq 0$ or $b \neq 0$. Suppose d and d' satisfied the properties in the definition of the greatest common divisor of a and b. In particular, both d and d' are common divisors of a and b. Since d is a greatest common divisor of a and b, $d' \leq d$. Similarly since d' is a greatest common divisor, $d \leq d'$. Alas, $d = d'$. ■

Exercise.

1. Compute $\gcd(a, b)$ for the following pairs of integers.

    (a) $a = -1$, $b = 27$.

    (b) $a = 8$, $b = 13$.

    (c) $a = 13$, $b = 21$.

    (d) $a = 21$, $b = 34$.

    (e) $a = 3427$, $b = 20184$.

2. Suppose $\gcd(a, b) = d$. What is $\gcd(a^2, b^2)$?

In Question 1 of the previous exercise, for all but the last pair of integers, it's easy enough to simply write out the divisors of both integers and look for the biggest one. However, this quickly becomes tedious. Thankfully we have the Euclidean Algorithm at our disposal. Let's remind ourselves how it works with an example, and then we'll prove that it does indeed work as advertised.

Example. Let's apply the Euclidean Algorithm to compute $\gcd(203, 77)$. We repeatedly apply the division algorithm as follows:

$$203 = 2 \cdot 77 + 49$$
$$77 = 1 \cdot 49 + 28$$
$$49 = 1 \cdot 28 + 21$$
$$28 = 1 \cdot 21 + 7$$
$$21 = 3 \cdot 7 + 0.$$

The last non-zero remainder gives us that $\gcd(203, 77) = 7$.

Let's write down this algorithm in general.

The Euclidean Algorithm:

Let a and b be positive integers with $a > b$ (if $a = b$ then $\gcd(a, b) = a$, so we don't need to worry about this case).

Set $r_0 = a$, $r_1 = b$, and define r_n for $n \geq 2$ as follows. Suppose r_{n-2} and r_{n-1} are positive integers with $r_{n-2} > r_{n-1}$. By the division algorithm, there exists integers q_{n-1} and r_n, with $0 \leq r_n < r_{n-1}$ so that $r_{n-2} = q_{n-1}r_{n-1} + r_n$. We now have $b > r_2 > r_3 \cdots \geq 0$, so the sequence $r_2, r_3, \ldots$ must become zero (after at most b steps). Let r_m be the last non-zero element in the sequence. The algorithm looks like this:

$$a = q_1 b + r_2$$
$$b = q_2 r_2 + r_3$$
$$r_2 = q_3 r_3 + r_4$$
$$\vdots$$
$$r_{m-2} = q_{m-1} r_{m-1} + r_m$$
$$r_{m-1} = q_m r_m + 0.$$

Then $\gcd(a, b) = r_m$.

To prove this works, we first have an important lemma.

Lemma 5. *If $a, q, b, r \in \mathbb{Z}$, $a \neq 0$ or $b \neq 0$, are such that $a = qb + r$, then $\gcd(a, b) = \gcd(b, r)$.*

Proof. If we show that the set of common divisors of a and b is equal to the set of divisors of b and r, then we can conclude they must have the same greatest common divisor. To this end, suppose $c \mid a$ and $c \mid b$. Then $c \mid a - qb = r$, so c is a common divisor of b and r. Conversely, if $c \mid b$ and $c \mid r$, then $c \mid qb + r = a$, so c is a common divisor of a and b. Alas, c is a common divisor of a and b if and only if c is a common divisor of b and r, completing the proof. ∎

Applying Lemma 5 to the Euclidean Algorithm we get

$$\gcd(a, b) = \gcd(b, r_2) = \gcd(r_2, r_3) = \gcd(r_3, r_4) = \cdots = \gcd(r_{m-1}, r_m) = \gcd(r_m, 0) = r_m$$

which proves that the Euclidean algorithm works!

Exercise. Find the greatest common divisor of the year you were born and the year one of your parents was born.

1.2 Linear Diophantine equations and Bezout

There will be several serious exploitations of the Euclidean algorithm, here's the first, and it allows us to prove seemingly difficult things about greatest common divisors rather easily. In fact, it will have far reaching consequences throughout the rest of the course.

Bezout's identity is a way to express $\gcd(a, b)$ in terms of a and b. For example, $\gcd(3, 5) = 1$ and we have $3 \cdot 2 - 5 \cdot 1 = 1$. Although this is just one example, this kind of thing happens in general. Even better, the Euclidean algorithm tells us how to write the greatest common divisor of a and b as $ax + by$ for some integers x and y!

Theorem 6 (Bezout's identity). *Let $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$. Then there exist integers x and y such that $\gcd(a, b) = ax + by$.*

Before proving this wonderful fact, let's run through an example with it.

Example (Dilcue). Recall we computed $\gcd(203, 77) = 7$ via the Euclidean algorithm as follows:

$$\begin{aligned}
203 &= 2 \cdot 77 + 49 \\
77 &= 1 \cdot 49 + 28 \\
49 &= 1 \cdot 28 + 21 \\
28 &= 1 \cdot 21 + 7 \\
21 &= 3 \cdot 7 + 0.
\end{aligned}$$

Now we run Dilcue, with is just the above algorithm backwards, starting with the second last line and substituting our way up the series of equations.

$$\begin{aligned}
7 &= 28 - 1 \cdot 21 \\
&= 28 - 1(49 - 28) \\
&= 2 \cdot 28 - 1 \cdot 49 \\
&= 2 \cdot (77 - 49) - 1 \cdot 49 \\
&= 2 \cdot 77 - 3 \cdot 49 \\
&= 2 \cdot 77 - 3(203 - 2 \cdot 77) \\
&= 8 \cdot 77 - 3 \cdot 203.
\end{aligned}$$

Therefore we have $7 = 77(8) - 203(3)$.

Euclid followed by Dilcue gives us a way to find integers x and y so that $\gcd(a,b) = ax + by$. Let's turn this example into a proof of Bezout's identity.

Proof of Bezout's identity. Suppose $a > b > 0$ (we will deal with all other cases at the end). Perform the Euclidean algorithm, and rearranging the second-last equation gives

$$\gcd(a,b) = r_m = r_{m-2} - q_{m-1}r_{m-1}$$

so r_m is an integer combination of r_{m-1} and r_{m-2}.

Claim. If $\gcd(a,b)$ is written as an integer combination of r_{k-2} and r_{k-1} for some $k \le m$, then $\gcd(a,b)$ can be written as an integer combination of r_{k-2} and r_{k-3}.

Proof of claim. Suppose $\gcd(a,b) = ur_{k-1} + vr_{k-2}$ for some integers u and v. Then the $(k-2)$nd line of the Euclidean algorithm can be rearranged to give

$$r_{k-1} = r_{k-3} - q_{k-2}r_{k-2}.$$

Substituting gives

$$\gcd(a,b) = u(r_{k-3} - q_{k-2}r_{k-2}) + vr_{k-2} = ur_{k-3} + (v - uq_{k-2})r_{k-2}$$

completing the proof of the claim. ∎

Now, applying the claim repeatedly gives us that $\gcd(a,b)$ can be written as an integer combination of $r_0 = a$ and $r_1 = b$, that is, $\gcd(a,b) = ax + by$ for some integers x, y.

Now, if a or b is negative, we can simply replace a and b by $|a|$ and $|b|$, find a solution to $\gcd(a,b) = |a|\, x + |b|\, y$, and simply change the sign of x or y to get a solution to $\gcd(a,b) = ax + by$. If $a < b$, we can switch the roles of x and y.

If $|a| = |b|$, then the equation we wish to solve takes the form $|a| = a(x \pm y)$ (since $a = \pm b$). This equation can be solved with $x = 1$ and $y = 0$ if a is positive, and $x = -1$ and $y = 0$ if a is negative. ∎

An important thing about the proof of Bezout's identity is that although an explicit formula for how to come up with a solution to the equation $\gcd(a,b) = ax + by$ is not provided, it does give us a recipe for coming up with the solution. And that recipe is exactly to do the Euclidean Algorithm forwards, and then backwards. Such a proof is called a *constructive proof* in mathematics.

Bezout's identity turns out to be super useful! From the definition of the greatest common divisor of a and b, we know that any other divisor c is such that $c \le \gcd(a,b)$. We can now prove something even stronger.

Proposition 7. *Let $a,b \in \mathbb{Z}$, $a \ne 0$ or $b \ne 0$. We have $c \mid a$ and $c \mid b$ if and only if $c \mid \gcd(a,b)$.*

Proof. Suppose $c \mid \gcd(a,b)$. Notice that $\gcd(a,b) \mid a$, so since $c \mid \gcd(a,b)$ and since divisibility is transitive, $c \mid a$. Similarly $c \mid b$.

Conversely, suppose $c \mid a$ and $c \mid b$. Then by Bezout's identity, there are integers $x, y \in \mathbb{Z}$ so that $ax + by = \gcd(a,b)$. We have $c \mid ax + by$, so $c \mid \gcd(a,b)$. ∎

Bezout's identity gives us a solution to a very specific equation. Such equations are called **linear Diophantine equations**. A Linear Diophantine equation is an equation of the form

$$d = ax + by$$

where $a, b, d \in \mathbb{Z}$.

As an astute reader, you may have recognised $d = ax + by$ as the equation of a line (in fact the line has slope $\frac{-a}{b}$ if $b \neq 0$, and is vertical if $b = 0$). So if we allow x and y to be real numbers, there is always a solution to any linear Diophantine equation (just choose x and y so that (x, y) is a point on the line). However, over the integers, things aren't as simple!

Example. The linear Diophantine equation $4x + 10y = 3$ does not have a solution over $\mathbb{Z}$ since the left hand side is even and the right hand side is odd.

Geometrically, asking for an integer solution to $ax + by = d$ is asking for a point (x, y) on the line with integer coordinates! A point $(x, y) \in \mathbb{R}^2$ where both x and y are integers is called a **lattice point**.

Thanks to Bezout, we are now in a position to work out exactly when a linear Diophantine equation has an integer solution, or equivalently, when a line of the form $ax + by = d$ passes through a lattice point in $\mathbb{R}^2$.

Theorem 8. *Let $a, b \in \mathbb{Z}$, $a \neq 0$ or $b \neq 0$. There exist integers $x, y \in \mathbb{Z}$ such that $ax + by = d$ if and only if $\gcd(a, b) \mid d$.*

Proof. Suppose there are integers x, y so that $ax + by = d$. Since $\gcd(a, b) \mid a$ and $\gcd(a, b) \mid b$, then $\gcd(a, b) \mid ax + by$ so $\gcd(a, b) \mid d$. Conversely, suppose $k \gcd(a, b) = d$ for some $k \in \mathbb{Z}$. Then by Bezout's identity, there exist $x, y \in \mathbb{Z}$ so that $\gcd(a, b) = ax + by$. Then $d = k \gcd(a, b) = a(kx) + b(ky)$, completing the proof. ∎

Exercise.

1. Solve the linear Diophantine equation $203x + 77y = 49$ over $\mathbb{Z}$.

2. Find infinitely many integer solutions to the Linear diophantine equation $4x + 10y = 16$. Have you found them all?

Lecture 5 - 15/01

1.3 Coprimeness

The case when $\gcd(a, b) = 1$ is special, and worth investigating some more. After all, when you write a fraction $\frac{a}{b}$ in lowest terms, this exactly means $\gcd(a, b) = 1$.

Definition. We say integers a and b are **coprime** or **relatively prime** if $\gcd(a, b) = 1$.

Exercise. True or False: If a and b are coprime and b and c are coprime, then a and c are coprime.

When attempting to solve a linear Diophantine equation $ax + by = d$, having that $\gcd(a, b) = 1$ is a dream! Since $1 \mid a$ for all $a \in \mathbb{Z}$, Theorem 8 gives us that $ax + by = d$ always has a solution, regardless of what d is! Let's see what else we can squeeze out of coprimeness.

Theorem 9. *Two integers a and b are coprime if and only if there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.*

Proof. This is an exercise. ∎

As with any if and only if statement in mathematics, we now have that an equivalent definition of a and b being coprimes is that there is an integer solution to $ax + by = 1$. For example, we know $201 - 25(8) = 1$, so we immediately have $\gcd(201, 25) = 1$.

Here are some useful facts about coprime integers.

Proposition 10. *Let a and b be coprime integers.*

1. *If $a \mid c$ and $b \mid c$, then $ab \mid c$.*

2. *If $a \mid bc$ then $a \mid c$.*

Proof.   1. We know there exist $x, y \in \mathbb{Z}$ so that $ax + by = 1$. Since $ak = bl = c$ for some $k, l \in \mathbb{Z}$ we have
$$c = cax + cby = ablx + abky = ab(lx + ky)$$
so $ab \mid c$.

2. As in part 1, there exist $x, y \in \mathbb{Z}$ so that $cax + cby = c$. Since $a \mid a$ and $a \mid bc$ we have $a \mid cax + cby$ so $a \mid c$. ∎

Exercise. Find integers a, b, c ($a \neq 0$ or $b \neq 0$) so that:

1. $a \mid c$ and $b \mid c$ but $ab \nmid c$.

2. $a \mid bc$ but $a \nmid c$.

2   Prime numbers

Prime numbers are the building blocks of the integers. They are fundamental, and mysterious. As you will see, we understand some things about them (like that there are infinitely many primes), but there are some basic questions to which we do not know the answer (such as whether or not there are infinitely many pairs of primes of the form p and $p + 2$).

Definition. An integer $p > 1$ is **prime** if its only positive divisors are 1 and p. An integer is **composite** otherwise.

Appendix A has a list of the first 1000 primes. Please let me know if you find any patterns. Let's begin our investigation into primes.

Observe that $4 \mid 60$ and $60 = 6 \cdot 10$. However, $4 \nmid 6$ and $4 \nmid 10$. This may seem suprising, but what's going on here is that since $2 \mid 6$ and $2 \mid 10$ when we combine them we get $2 \cdot 2 \mid 6 \cdot 10$, so we have $4 \mid 10$. So, the fact that you can have $4 \mid ab$ but $4 \nmid a$ and $4 \nmid b$ is a consequence of the fact that 4 is composite. So, maybe, we shouldn't expect this kind of thing to happen with primes, and indeed it doesn't!

Here is an important observation: If p is prime, its only positive divisors are 1 and p. Therefore $\gcd(p, a)$ is either 1 or p. So, if $p \nmid a$, then $\gcd(p, a) = 1$.

Proposition 11. *Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Proof. If $p \mid a$ then we're done, so suppose $p \nmid a$. Since p is not a factor of a, we must have that p and a are coprime. Therefore $p \mid b$ by Proposition 10. ∎

Exercise.

- Prove that if $p \mid a_1 \cdots a_t$ for some $a_1, \ldots, a_t \in \mathbb{Z}$, then $p \mid a_i$ for some i.

- Prove that if p is prime and $p \mid a^k$ for some integer $k \geq 0$, then $p^k \mid a^k$.

Lecture 6 - 17/01

2.1   The Fundamental Theorem of Arithmetic

We are now ready to prove a very fundamental theorem about numbers, called, well, the Fundamental Theorem of Arithmetic. It really shows why we consider prime numbers to be the building blocks of the integers.

Theorem 12 (Fundamental Theorem of Arithmetic)**.** *Each integer $n > 1$ has a prime-power factorisation*

$$n = p_1^{e_1} \cdots p_k^{e_k}$$

where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers. Furthermore, other than permuting the factors, this factorisation is unique.

For example, $100 = 5^2 \cdot 2^2$, which we could also write as $100 = 2^2 \cdot 5^2$ or $100 = 2 \cdot 5^2 \cdot 2$. However, whichever way we write 20 as a product of primes, it must contain two 2s and two 5s.

Let's prove the fundamental theorem of arithmetic!

Proof. For existence we will proceed by induction on n. The base case $n = 2$ is easy, since $n = 2^1$. Now suppose $n > 2$ and every integer k such that $2 \leq k \leq n - 1$ has a prime factorisation. If n is prime, then $n = n^1$ and we're done. If n is composite, then $n = ab$ for some integers $2 \leq a, b \leq n - 1$. Therefore a and b have prime factorisations, so $n = ab$ has a prime factorisation (by multiplying together the prime factorisations of a and b).

For uniqueness, we will again induct on n. The base case $n = 2$ is an exercise for the reader, to show that the only prime factorisation of 2 is indeed 2^1. Now, suppose every integer k such that $2 \leq k \leq n - 1$ has a unique prime factorisation (up to permuting the factors). Suppose $n = p_1^{e_1} \cdots p_s^{e_s} = q_1^{f_1} \cdots q_t^{f_t}$ where the p_i are distinct primes, the q_i are distinct primes and the e_i and f_i are positive integers. Since $p_1 \mid q_1^{f_1} \cdots q_t^{f_t}$, we have $p_1 \mid q_j$ for some j. By permuting the q_i, we have $p_1 \mid q_1$ and since q_1 is prime we have $p_1 = q_1$. Then $\frac{n}{p_1} = p_1^{e_1 - 1} p_2^{e_2} \cdots p_s^{e_s} = q_1^{f_1 - 1} q_2^{f_2} \cdots q_t^{f_t}$. However, $2 \leq \frac{n}{p_1} \leq n - 1$ so these two factorisations of $\frac{n}{p_1}$ are the same after permuting the factors. Therefore the original two factorisations of n are the same after the permuting the factors. ∎

Great! The fundamental theorem of arithmetic turns out to be outrageously useful for studying divisibility of integers (among other things). Let's take a look at an example.

Example. Let's look at the prime factorisations of all the positive divisors of $2^3 \cdot 3^2 \cdot 5 = 360$. They

are

$$1 = 2^0 \cdot 3^0 \cdot 5^0$$
$$2 = 2^1 \cdot 3^0 \cdot 5^0$$
$$4 = 2^2 \cdot 3^0 \cdot 5^0$$
$$8 = 2^3 \cdot 3^0 \cdot 5^0$$
$$3 = 2^0 \cdot 3^1 \cdot 5^0$$
$$6 = 2^1 \cdot 3^1 \cdot 5^0$$
$$12 = 2^2 \cdot 3^1 \cdot 5^0$$
$$24 = 2^3 \cdot 3^1 \cdot 5^0$$
$$9 = 2^0 \cdot 3^2 \cdot 5^0$$
$$18 = 2^1 \cdot 3^2 \cdot 5^0$$
$$36 = 2^2 \cdot 3^2 \cdot 5^0$$
$$72 = 2^3 \cdot 3^2 \cdot 5^0$$
$$5 = 2^0 \cdot 3^0 \cdot 5^1$$
$$10 = 2^1 \cdot 3^0 \cdot 5^1$$
$$20 = 2^2 \cdot 3^0 \cdot 5^1$$
$$40 = 2^3 \cdot 3^0 \cdot 5^1$$
$$15 = 2^0 \cdot 3^1 \cdot 5^1$$
$$30 = 2^1 \cdot 3^1 \cdot 5^1$$
$$60 = 2^2 \cdot 3^1 \cdot 5^1$$
$$120 = 2^3 \cdot 3^1 \cdot 5^1$$
$$45 = 2^0 \cdot 3^2 \cdot 5^1$$
$$90 = 2^1 \cdot 3^2 \cdot 5^1$$
$$180 = 2^2 \cdot 3^2 \cdot 5^1$$
$$360 = 2^3 \cdot 3^2 \cdot 5^1$$

You may notice something about all the prime factorisations - the exponents of the primes are bounded above by the exponents of the primes in the original factorisation. To state the next proposition slickly, we are going to allow exponents of zero on the prime factorisations. This is just so we can take any two numbers and write them as a product of powers of the same set of primes. For example, $6 = 2^1 \cdot 3^1 \cdot 5^0$ and $15 = 2^0 \cdot 3^1 \cdot 5^1$.

Proposition 13. *Let $n = p_1^{e_1} \cdots p_k^{e_k}$ and $m = p_1^{f_1} \cdots p_k^{f_k}$, where the p_i are distinct primes and $e_i, f_i \geq 0$ for all i. Then $m \mid n$ if and only if $f_i \leq e_i$ for all i.*

Proof. This is an exercise. ∎

In fact, we can write down formulas for a whole bunch of things in terms of the prime factorisations.

Proposition 14. *Let $a = p_1^{e_1} \cdots p_k^{e_k}$ and $b = p_1^{f_1} \cdots p_k^{f_k}$, where the p_i are distinct primes and*

$e_i, f_i \geq 0$ *for all i. Then*

$$ab = p_1^{e_1+f_1} \cdots p_k^{e_k+f_k}$$
$$a/b = p_1^{e_1-f_1} \cdots p_k^{e_k-f_k}$$
$$\gcd(a,b) = p_1^{\min\{e_1,f_1\}} \cdots p_k^{\min\{e_k,f_k\}}.$$

Proof. The proof of this proposition is also an exercise. ∎

Example. Suppose p,q, and r are distinct primes. Then $\gcd(pq^2, qr^2) = q$.

The formula $\gcd(a,b) = p_1^{\min\{e_1,f_1\}} \cdots p_k^{\min\{e_k,f_k\}}$ certainly seems like an easy way to compute the greatest common divisor of two integers, and it is! Provided you already have the prime factorisations of the integers in which you are interested. An attractive feature of the Euclidean algorithm is that you don't need the prime factorisations of a and b to compute $\gcd(a,b)$. An in general, it's rather difficult to compute the prime factorisation of an integer.

Let's push the fundamental theorem of arithmetic a little further.

Lecture 7 - 20/01

Proposition 15. *Let $a = p_1^{e_1} \cdots p_k^{e_k}$ where the p_i are distinct primes and e_i is a positive integer for all i. Then a is a perfect square if and only if e_i is even for all i.*

Proof. Suppose $a = b^2$ and the prime factorisation of b is given by $b = p_1^{f_1} \cdots p_k^{f_k}$. Then $a = b^2 = p_1^{2f_1} \cdots p_k^{2f_k}$. Since the prime factorisation of a is unique up to changing the order of the primes, the exponent of each prime is even. Conversely, suppose the exponent on each prime in the prime factorisation of a is even. Then $a = p_1^{2f_1} \cdots p_k^{2f_k} = (p_1^{f_1} \cdots p_k^{f_k})^2$, completing the proof. ∎

Let's keep going, this time to construct infinitely many irrational numbers.

Proposition 16. *If a positive integer m is not a perfect square, then $\sqrt{m}$ is irrational.*

Proof. Suppose $\sqrt{m} = \frac{a}{b}$ is rational, so a and b are integers. We may choose the positive square root and therefore assume a and b are positive integers. Consider the prime factorisations $a = p_1^{e_1} \cdots p_k^{e_k}$ and $b = p_1^{f_1} \cdots p_k^{f_k}$ where the p_i are distinct primes and the e_i and f_i are non-negative integers for all i. Then $m = \frac{a^2}{b^2}$ is an integer so $b^2 \mid a^2$. Therefore $2f_i \leq 2e_i$ for all i and we have $m = p_1^{2(e_i-f_i)} \cdots p_k^{2(e_k-f_k)}$. Since each exponent in the prime factorisation of m is even, m is a perfect square. ∎

So, for example, $\sqrt{24}$ is irrational!

Exercise. Let a and n be positive integers. When is $a^{\frac{1}{n}}$ rational?

Finally, we finish with one of my favourite proofs of all time. A proof of the infinitude of primes.

Theorem 17. *There are infinitely many primes.*

Proof. Suppose towards a contradiction that there are finitely many primes $p_1, \ldots, p_k$. Consider the integer $m = p_1 \cdots p_k + 1$. Then for every i, since $p_i > 1$, by the division algorithm, m has remainder 1 when divided by p_i. Thus $p_i \nmid m$ for all i. However, m has some prime factor q by the Fundamental Theorem of Arithmetic, contradicting our assumption that $p_1, \ldots, p_k$ is a complete list of primes. ∎

Neat!

3 Continued fractions

Let's do something completely different, the Euclidean algorithm. Here is the Euclidean algorithm on the pair of integers 203, 77.

$$203 = 2 \cdot 77 + 49$$
$$77 = 1 \cdot 49 + 28$$
$$49 = 1 \cdot 28 + 21$$
$$28 = 1 \cdot 21 + 7$$
$$21 = 3 \cdot 7 + 0.$$

But, I want to write this slightly differently:

$$\frac{203}{77} = 2 + \frac{49}{77}$$
$$\frac{77}{49} = 1 + \frac{28}{49}$$
$$\frac{49}{28} = 1 + \frac{21}{28}$$
$$\frac{28}{21} = 1 + \frac{7}{21}$$
$$\frac{21}{7} = 3 + \frac{0}{7}.$$

In fact, we can write this whole process in a curious looking form called a continued fraction as follows.

The first equation can be rewritten as

$$\frac{203}{77} = 2 + \frac{1}{\left(\frac{77}{49}\right)}$$

at which point the second equation comes in to the picture and gives

$$\frac{203}{77} = 2 + \frac{1}{1 + \frac{28}{49}}$$
$$= 2 + \frac{1}{1 + \frac{1}{\left(\frac{49}{28}\right)}}.$$

Of course, now the third equation comes in, yielding

$$\frac{203}{77} = 2 + \frac{1}{1 + \frac{1}{1 + \frac{21}{28}}}.$$

Continuing in this fashion through the entire Euclidean algorithm gives us the **continued fraction expansion** of $\frac{203}{77}$

$$\frac{203}{77} = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}}.$$

This is great, but who cares? I mean, it's fun to write down nested fractions like this and all. But at some point, I'm going to be too lazy to write down continued fractions all the way to their completion. Let's see what number comes out if I just decide to stop after some point.

Let's first note that $\frac{203}{77} \approx 2.63636363....$ Here is what comes out of our truncated continued fractions.

$$2 = 2$$

$$2 + \frac{1}{1} = 3$$

$$2 + \frac{1}{1 + \frac{1}{1}} = \frac{5}{2} = 2.5$$

$$2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{8}{3} = 2.666...$$

$$2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}} = 2.636363636363...$$

Lecture 8 - 22/01

Cool! Each successive term appears to be a better and better approximation to our original fraction. Let's try something else, and let's run the Euclidean algorithm on π (stay with me here).

Here are the first few steps of what actually goes on forever (becuse π is irrational).

$$\pi = 3 + r_1$$

$$\frac{1}{r_1} = 7 + r_2$$

$$\frac{1}{r_2} = 15 + r_3$$

$$\frac{1}{r_3} = 1 + r_4$$

$$\frac{1}{r_4} = 292 + r_5$$

$$\frac{1}{r_5} = 1 + r_6.$$

And in fact, the integer parts that appear have been well studied, and here are the first few

$$3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2.$$

So, let's play the game we did above, truncating our continued fraction and seeing what comes out.

Here are the first few.

$$3 = 3$$

$$3 + \frac{1}{7} = \frac{22}{7} = 3.142857142857...$$

$$3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106} = 3.141509433...$$

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113} = 3.1415929203...$$

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}} = \frac{103993}{33102} = 3.14159265301...$$

These are all very good approximations of π, and in some sense, they are the best possible approximations for fractions with denominators as large as they are.

Before we continue, we will need some notation. To save us writing out the continued fraction in full each time, we have the following short-hand. For integers $a_0, \ldots, a_n$ denote the continued fraction by

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{1}{a_n}}}}} = [a_0, a_1, a_2, \ldots, a_n].$$

So, we have

$$\frac{203}{77} = \frac{29}{11} = [2, 1, 1, 1, 3]$$

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, ...]$$

and note that since π is irrational, the continued fraction continues (although not in any percievable pattern).

Exercise. Prove that the continued fraction expansion of a real number x is finite if and only if $x \in \mathbb{Q}$.

Exercise. Compute the continued expansion expansion of e.

Let's compute the continued fraction expansion of $\sqrt{3}$. This time, instead of using a computer to figure out the integer part, we will keep all our computations in exact form, and see how far we can get.

We know $\sqrt{3}$ is between 1 and 2. So the first line in our computation will be $\sqrt{3} = 1 + (\sqrt{3} - 1)$. Then we flip the fraction part, and repeat. By rationalising the denominator we have $\frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}$. Since $\sqrt{3} + 1$ is between 2 and 3, we know $\frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2}$. Again, flipping the fractional part and rationalising the denominator gives $\frac{2}{\sqrt{3}-1} = \sqrt{3} + 1$, which we know is between 2 and 3. This gives $\sqrt{3} + 1 = 2 + (\sqrt{3} - 1)$. Hang on, I swear we've been here before! We know what happens

when we get to a fractional part of $\sqrt{3} - 1$. Taking this rambling paragraph and organising it gives

$$\sqrt{3} = 1 + (\sqrt{3} - 1)$$

$$\frac{1}{\sqrt{3} - 1} = 1 + \frac{\sqrt{3} - 1}{2}$$

$$\frac{2}{\sqrt{3} - 1} = 2 + (\sqrt{3} - 1)$$

$$\frac{1}{\sqrt{3} - 1} = 1 + \frac{\sqrt{3} - 1}{2}$$

$$\frac{2}{\sqrt{3} - 1} = 2 + (\sqrt{3} - 1)$$

$$\frac{1}{\sqrt{3} - 1} = 1 + \frac{\sqrt{3} - 1}{2}$$

$$\frac{2}{\sqrt{3} - 1} = 2 + (\sqrt{3} - 1)$$

$$\vdots$$

Therefore the continued fraction expansion of $\sqrt{3}$ is

$$\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, 1, 2, \ldots] = [1, \overline{1, 2}].$$

Exercise. Compute the continued fraction expansion of $\sqrt{n}$ for $n = 5, 6, 7, 8, 10$.

Exercise. Find an irrational number x with continued fraction expansion $[1, 1, 1, 1, 1, \ldots]$.

Lecture 9 - 24/01

3.1 Convergents and approximating irrationals

As we saw in the introduction to this chapter, the truncated continued fractions for π appear to approximate π very well, and the longer we make our truncated continued fraction, the better the approximation. These truncated continued fractions are called convergents.

Definition. Let x be an irrational number and $x = [a_0, a_1, \ldots]$ its continued fraction expansion. The nth convergent of x is the rational number

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

So, the 0th, 1st, 2nd, 3rd, and 4th convergents of π, which we computed above, are 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, and $\frac{103993}{33102}$ respectively. Let's compute the first 4 convergents of $\sqrt{3}$. We have

$$\frac{p_0}{q_0} = 1$$

$$\frac{p_1}{q_1} = 1 + \frac{1}{1} = 2$$

$$\frac{p_2}{q_2} = 1 + \frac{1}{1 + \frac{1}{2}} = \frac{5}{3}$$

$$\frac{p_3}{q_3} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1}}} = \frac{7}{4}.$$

19

The decimal expansion for $\sqrt{3}$, up to 10 decimal places, is

$$\sqrt{3} = 1.7320508075...$$

That's pretty close to the 3rd convergent, which is $\frac{7}{4} = 1.75$. This is pretty unimpressive though, since I could have come up with this approximation without the Euclidean algorithm! But, let's look at the 7th convergent:

$$\frac{p_7}{q_7} = \frac{97}{56} = 1.732142857...$$

which is equal to $\sqrt{3}$ when both are rounded to 4 decimal places! This is pretty impressive, because if you asked me to approximate $\sqrt{3}$ to 4 decimal places with a rational number I would have chosen $\frac{17321}{10000}$. My approximation is pretty good, but the denominator has 5 digits. The denominator of the 7th convergent only has 2 digits! In a very precise sense, the convergents give rise to the best approximations of $\sqrt{3}$.

Definition. Let x be a real number. The rational number $\frac{p}{q}$ (written with $\gcd(p, q) = 1$) is a **best rational approximation** to x if

$$\left| x - \frac{p}{q} \right| = \min \left\{ \left| x - \frac{p'}{q'} \right| : q' \leq q \right\}.$$

Here is the main theorem about how much convergents, well, converge.

Theorem 18. *For any real number x, the convergents of x are best approximations of x.*

We won't prove this in these notes, but it is a good thing to think about.

So, for example, if you wanted a better rational approximation of $\sqrt{3}$ than $\frac{97}{56}$, you would need a fraction with denominator at least 57.

Exercise. Let $\phi = \frac{1+\sqrt{5}}{2}$. Find the best rational approximation $\frac{p}{q}$ of ϕ such that $q \leq 377$.

Exercise. Let c_n be the nth convergent of an irrational number x. Prove that $\{c_n\}$ is a Cauchy sequence, and so the limit $\lim_{n \to \infty} c_n$ exists. What is the limit?

So we know convergents give best approximations. What about the other way around? Ordinarily when I ask a question like this, it's because the answer is yes. This time, the answer is no. Mathematics is often beautiful and the theorems are often neat, but just because a statement is beautiful and neat, doesn't mean it's true.

Let's look at the first few convergents of π again.

$$\frac{p_0}{q_0} = 3 = 3$$

$$\frac{p_1}{q_1} = \frac{22}{7} = 3.142857142857...$$

$$\frac{p_2}{q_2} = \frac{333}{106} = 3.141509433...$$

$$\frac{p_3}{q_3} = \frac{355}{113} = 3.1415929203...$$

$$\frac{p_4}{q_4} = \frac{103993}{33102} = 3.14159265301...$$

The third convergent is accurate to 6 decimal places. The denominator in the fourth convergent has 5 digits. It seems unreasonable to think that there are no best approximations with denominator between 113 and 33102. Indeed this is unreasonable and there are lots of best approximations not in the list of convergents. In fact, here are the first few best approximations (listed in order of increasing denominator)

$$\frac{3}{1}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \frac{22}{7}.$$

Only the first and last of these are convergents of π.

This raises the question, if we have a best rational approximation of a number x, can we tell whether or not it's a convergent? Here are a couple of theorems which we won't prove, that partially answer this question.

Theorem 19 (Dirichlet's Approximation Theorem). *Let x be a real number, and $\frac{p}{q}$ a convergent of x. Then $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$.*

So, if you have a convergent with a 20-digit denominator, then it will be accurate to about 40 digits.

Theorem 20 (Legendre's Theorem). *If $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$, then $\frac{p}{q}$ is a convergent of x.*

Exercise. Show that the converses of Dirichlet's and Legendre's Theorems are false.

3.2 Pell's equation

One of the motivations for studying the Euclidean algorithm and number theory in general is that it is generally more difficult to solve an equation with integers, than it is with real numbers. We have seen this explicitly when solving linear Diophantine equations.

Let's see another curious example of trying to solve something over the integers. By now we are all familiar with square numbers: $1, 4, 9, 16, 25, \ldots$ These are called square numbers because, well, they tell us something about the number of things (say coins) you can put in a square.
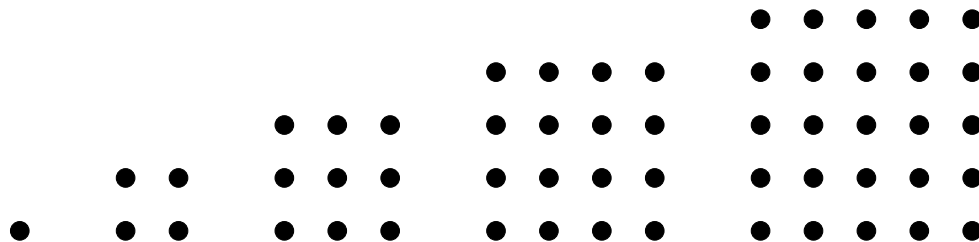
The nth square number is of course given by n^2. There is the lesser known, but just as pleasant to be around, cousin of the square number, the triangular number. These are the number of things you can arrange into a triangle with base n.
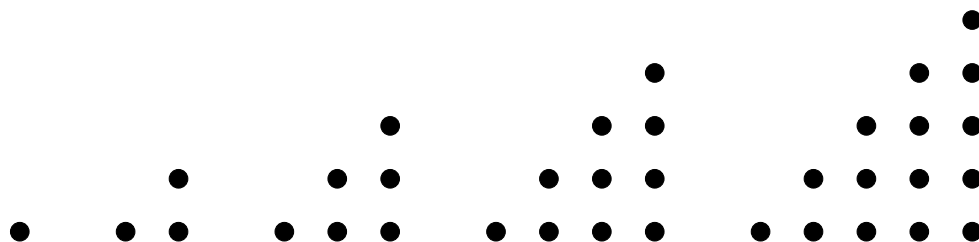
So the first few triangular numbers are $1, 3, 6, 10, 15$ and in fact the nth triangular number is the sum of the first n integers. Thus the nth triangular number is $\frac{n(n+1)}{2}$.

Now, here's a fun question. Can you find a number of dots that can be arranged into both a square, and a triangle? Well, let's list out the first few square numbers and triangular numbers and see if we can find anything.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
$\frac{n(n+1)}{2}$	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120

The blue numbers appear in both lists. The only ones, at least in the first 15 square and triangular numbers, are 1 and 36. Are there others?

Investigating this question further, we want to find integers m and n so that $m^2 = \frac{n(n+1)}{2}$. Rearranging a little gives $(2n+1)^2 = 8m^2 + 1$. If we let $x = 2n + 1$ and $y = 2m$ this gives

$$x^2 - 2y^2 = 1.$$

So if we're interested in square triangular numbers (I mean, who isn't?), we're interested in integer solutions to the equation $x^2 - 2y^2 = 1$. Since we are looking at x^2 and y^2, we may as well restrict our attention to $x, y \geq 0$ since we can generate other solutions by just flipping the signs of x and y.

Staring at this equation gives a couple of solutions right off the bat: $(x, y) = (1, 0)$ and $(x, y) = (3, 2)$. The first corresponds to the square-triangular number 0 (which I guess technically works), and the second to the number 1. But this can't be all the solutions, there must be one hiding in the weeds corresponding to the square-triangular number 36. The square number 36 gives $m = 6$ and the triangular number 36 gives $n = 8$. So, it should be the case that $(x, y) = (2n+1, 2m) = (17, 12)$ is another solution. A quick check: $17^2 - 2(12^2) = 289 - 2(288) = 1$. Phew!

So, let's focus on solving the equation $x^2 - 2y^2 = 1$, and those like it, and we'll return to the square triangular numbers later.

Lecture 10 - 27/01

Definition. Let $D > 0$ be an integer that is not a perfect square. **Pell's equation** is the equation $x^2 - Dy^2 = 1$.

Let's suppose we did have some solutions to Pell's equation $x^2 - Dy^2 = 1$. The left hand side is curious, and vaguely familiar. If $D = -1$ (which we have explicitly disallowed, but let's continue down the forbidden path anyway), the equation becomes $x^2 + y^2 = 1$, the equation of the unit circle. More importantly, it's the equation satisfied by the complex number $z = x + yi$, if $|z| = 1$. We know in $\mathbb{C}$, that $|z| \, |w| = |zw|$. Therefore, turning solutions to $x^2 + y^2 = 1$ into complex numbers, and multiplying them in $\mathbb{C}$, will yield more solutions! Maybe, if we're lucky, the same kind of thing happens with solutions to $x^2 - Dy^2 = 1$, and solutions correspond to some kinds of numbers with some kind of norm being 1. Let's try this with our couple of solutions to $x^2 - 2y^2 = 1$.

We know $(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (17, 12)$ are solutions. Let's pretend these are coordinates of numbers that are kind of like the complex numbers, except with D playing the role of -1. Then the corresponding numbers would be $3 + 2\sqrt{2}$ and $17 + 12\sqrt{2}$. Let's multiply these together and see what happens.

$$(3 + 2\sqrt{2})(17 + 12\sqrt{2}) = 51 + (34 + 36)\sqrt{2} + 48 = 99 + 70\sqrt{2}.$$

Now the moment of truth, does $(x_3, y_3) = (99, 70)$ give us a solution to the Pell equation? We have

$$99^2 - 2(70^2) = 9801 - 2(4900) = 1.$$

Amazing! This gives another square triangular number, which is 1225.

Lemma 21. *Suppose (x_1, y_1) and (x_2, y_2) are solutions to Pell's equation $x^2 - Dy^2 = 1$. Then if $x_3 + y_3\sqrt{D} = (x_1 + y_1\sqrt{D})(x_2 + y_2\sqrt{D})$, then (x_3, y_3) is a solution.*

Proof. We have

$$(x_1 + y_1\sqrt{D})(x_2 + y_2\sqrt{D}) = (x_1 x_2 + Dy_1 y_2) + (x_1 y_2 + x_2 y_1)\sqrt{D}$$

so let $x_3 = x_1 x_2 + Dy_1 y_2$ and $y_3 = x_1 y_2 + x_2 y_1$. Then

$$
\begin{aligned}
x_3^2 - Dy_3^2 &= (x_1 x_2 + Dy_1 y_2)^2 - D(x_1 y_2 + x_2 y_1)^2 \\
&= x_1^2 x_2^2 + 2Dx_1 x_2 y_1 y_2 + D^2 y_1^2 y_2^2 - Dx_1^2 y_2^2 - 2Dx_1 x_2 y_1 y_2 - Dx_2^2 y_1^2 \\
&= (x_1^2 - Dy_1^2)(x_2^2 - Dy_2^2) \\
&= 1
\end{aligned}
$$

completing the proof. ∎

Cool, so if we have a solution to Pell's equation, we can potentially find a bunch more! Let's do this with our solutions that we already have to $x^2 - 2y^2 = 1$. We know $(3, 2)$ is a solution, so let's just take powers of $3 + 2\sqrt{2}$ and see what happens. Here is the result arranged in a helpful table (along with the corresponding square-triangular number).

k	$(3 + 2\sqrt{2})^k$	solution (x, y)	square-triangular number
1	$3 + 2\sqrt{2}$	$(3, 2)$	1
2	$17 + 12\sqrt{2}$	$(17, 12)$	36
3	$99 + 70\sqrt{2}$	$(99, 70)$	1225
4	$577 + 408\sqrt{2}$	$(577, 408)$	41616
5	$3363 + 2378\sqrt{2}$	$(3363, 2378)$	1413721

We can find infinitely many solutions this way, and thus infinitely many square-triangular numbers!

Exercise. Suppose (x_1, y_1) is a solution to Pell's equation $x^2 - Dy^2 = 1$, with $x_1, y_1 > 0$. Suppose $k, l > 0$. Prove that if $(x_1 + y_1\sqrt{D})^k = (x_1 + y_1\sqrt{D})^l$ then $k = l$. Conclude that if there is one integer solution (x_1, y_1) with $x_1, y_1 > 0$ to the equation $x^2 - Dy^2 = 1$, then there are infinitely many.

So you have infinitely many of something, the natural question is: Do you have all of them?

Theorem 22. *Let $D > 2$ be an integer that is not a perfect square. Then there exists positive integers x_0, y_0 solving Pell's equation $x^2 - Dy^2 = 1$.*

*Furthermore, let (x_1, y_1) be the positive solution with the smallest x value (called the **fundamental solution**). Then any other positive solution is of the form (x_k, y_k) where $x_k + y_k\sqrt{D} = (x_1 + y_1\sqrt{D})^k$ for some positive integer k.*

So, in the case $D = 2$ that we have been playing around with, you can check that there are no solutions with $x = 1$ or $x = 2$, so $(3, 2)$ is the fundamental solution. Therefore all other solutions are of the form (x_k, y_k) where $x_k + y_k\sqrt{2} = (3 + 2\sqrt{2})^k$.

Exercise. Prove that there are no integer solutions to $x^2 - 2y^2 = 1$ with $x = 1$ or $x = 2$.

We are lead, finally, to the last question we shall ask in this course. How do we find the fundamental solution to Pell's equation?

Lemma 23. *Suppose (a, b) is a solution to Pell's equation $x^2 - Dy^2 = 1$. Then $\frac{a}{b}$ is a convergent of $\sqrt{D}$.*

Proof. We have $a^2 - Db^2 = 1$. Implying $(a - b\sqrt{D})(a + b\sqrt{D}) = 1$ and thus

$$a - b\sqrt{D} = \frac{1}{a + b\sqrt{D}} > 0.$$

Therefore $a > b\sqrt{D}$. Now let's see how good an estimate $\frac{a}{b}$ is to $\sqrt{D}$. We have

$$\left| \sqrt{D} - \frac{a}{b} \right| = \frac{a - b\sqrt{D}}{b} = \frac{1}{b(a + b\sqrt{D})} < \frac{1}{b(2b\sqrt{D})} < \frac{1}{2b^2}.$$

Therefore by Legendre's theorem, $\frac{a}{b}$ is a convergent of $\sqrt{D}$. ∎

Finally, somewhere to look to find solutions. If we look back at our solutions when $D = 1$ we see each of the solutions do indeed give convergents of $\sqrt{2}$. In fact the 5 solutions listed in the table above are the 1st, 3rd, 5th, 7th, and 9th convergents respectively. That is,

$$\frac{p_1}{q_1} = \frac{3}{2}, \quad \frac{p_3}{q_3} = \frac{17}{12}, \quad \frac{p_5}{q_5} = \frac{99}{70}, \quad \frac{p_7}{q_7} = \frac{577}{408}, \quad \text{and} \quad \frac{p_9}{q_9} = \frac{3363}{2378}.$$

Intriguing.

If Theorem 22 is correct (which it is), we just have to run through the convergents of $\sqrt{D}$ until we find a solution! The first one we find will have the smallest x-value (which is an interesting thing to think about) and therefore it will be our fundamental solution!

Lecture 11 - 29/01

Example. Let's find all the solutions to Pell's equation $x^2 - 7y^2 = 1$. To find the fundamental solution, we first compute the continued fraction expansion of $\sqrt{7}$. It turns out to be $[2, \overline{1, 1, 1, 4}]$. Let's compute the first few convergents, and see what they give when plugging them into Pell's equation.

Convergent $\frac{p}{q}$	$p^2 - 7q^2$
$\frac{2}{1}$	-3
$\frac{3}{1}$	2
$\frac{5}{2}$	-3
$\frac{8}{3}$	1

We can stop here since we have found our fundamental solution $(x, y) = (8, 3)$. Then all solutions are of the form (x_k, y_k) where $x_k + y_k\sqrt{7} = (8 + 3\sqrt{7})^k$ for $k > 1$.

Sometimes, the fundamental solution can be quite large and hard to come by. Notably, the fundamental solution to $x^2 - 61y^2 = 1$ is $(x, y) = (1766319049, 226153980)$!

There is a beautiful geometric story to be told regarding the set of all solutions to Pell's equation, but that is a story for another time. To get a hint at the story, do the following exercise.

Exercise. Let C be the curve in $\mathbb{R}^2$ defined by $x^2 - 2y^2 = 1$. We know $(1,0)$ and $(3,2)$ lie on C.

1. Let T be the tangent line to C at $(3,2)$, and let L be a line parallel to T passing through $(1,0)$. The line L intersects C at another point (a,b). Find (a,b).

2. Let S be the line joining $(3,2)$ to (a,b) (where (a,b) is your solution to the previous question), and let L be a line parallel to S passing through $(1,0)$. The line L intersects C at another point (c,d). Find (c,d).

3. What do you notice about the points (a,b) and (c,d). Any conjectures?

4 Modular Arithmetic

Modular arithmetic, sometimes called clock arithmetic, is a way of doing mathematics while focusing only on remainders. For example, suppose you knew today was Wednesday (which it happens to be as I write this!). What day will it be 22 days from now? The answer is Thursday. A quick way to see this is to note that we only care about what the remainder of 22 is when divided by 7. After 21 days (which is a multiple of 7), it will be Wednesday again, so in 22 days it will be Thursday.

Similarly, we cn do arithmetic on a clock with 12 numbers (just like a regular clock). We will call this clock $\mathbb{Z}_{12}$. It looks like this:

$$
\begin{array}{ccc}
& 11 \;\; 0 \;\; 1 & \\
10 & & 2 \\
9 & & 3 \\
8 & & 4 \\
& 7 \;\; 6 \;\; 5 &
\end{array}
$$
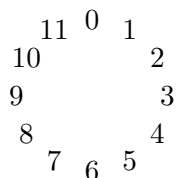
In $\mathbb{Z}_{12}$ we can do addition and multiplication as usual, except we identify some numbers (like 13 is the same as 1, and 12 is the same as 0). For example, $2 + 3 = 5$ and $7 + 8 = 3$ and $4 \cdot 5 = 8$. We will formalise all of this soon, so take these equations with a grain of salt (the statement $7 + 8 = 3$ taken out of context could get me fired!).

Let's do some more examples, this time in $\mathbb{Z}_7$:

$$
\begin{array}{ccc}
& 0 & \\
6 & & 1 \\
5 & & 2 \\
& 4 \quad 3 &
\end{array}
$$
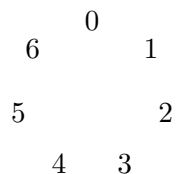
Here $2 + 1 = 3$, $-2 = 5$, and $3 \cdot 4 = 12 = 5$. What's really going on here is that we are associating all numbers that have a remainder of 3 when divided by 7, say, with the number 3 on the clock. Curiously, no harm seems to come to us if we do our arithmetic with different numbers that have the same remainder. For example, $5 + 4 = 9 = 2$, but we know $5 = -2$ and $4 = -3$. So we could instead have done $(-2) + (-3) = -5 = 2$. Either way, we get the same answer!

Let's do one more example, which is maybe a little more familiar to us, that of $\mathbb{Z}_2$. $\mathbb{Z}_2$ as a clock only has two things, 0 and 1. We can think of the 0 as all the numbers which have a remainder of 0 when divided by 2 (ie all the even numbers) and 1 as all the numbers that have a remainder of 1 when divided by 2 (ie all the odd numbers). Since there are only two things in $\mathbb{Z}_2$, it's not too onerous to draw out the addition and multiplication tables. Let's put them alongside tables which summarise what happens when you add and multiply odd and even numbers.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

+	even	odd
even	even	odd
odd	odd	odd

×	even	odd
even	even	even
odd	even	odd

Notice anything? The thing I'm trying to hint at is that the remainder of a and b when divided by 2 is all that matters when figuring out the remainder of $a + b$ and ab when divided by 2. Even better, in the previous sentence, I could have replaced 2 with n and it would still be true. The next section will be all about formalising and proving these ideas.

4.1   Congruences

The first step to formalising is to treat all integers which give the same remainder when divided by n as equal.

Definition. Let n be a positive integer and let a and b be any integers. We say a **is congruent to b modulo n** (or simply **mod n**), and write $a \equiv b \pmod{n}$, if $n \mid a - b$. If it is clear from context what n is, we may simply write $a \equiv b$.

You will find many different ways of denoting that a is congruent to b mod n in textbooks and online. Here are some: $a \equiv_n b$, $a \equiv b \mod n$, or $a \equiv b \mod (n)$. Sometimes, instead of saying a is congruent to b mod n, it is said that a is a **residue** of b mod n.

"But wait," I hear you object, "you lead us to believe remainders were important, not this strange n-divides-the-difference business!"

Well, I'm glad you brought that up.

Proposition 24. *Let n be a positive integer and $a, b \in \mathbb{Z}$. Then $a \equiv b \pmod{n}$ if and only if a and b have the same remainder when divided by n.*

Proof. Suppose $a \equiv b \pmod{n}$ and write $b = qn + r$ as per the division algorithm (so $0 \leq r < n$). Since $n \mid a - b$ we have $a = kn + b$ for some $k \in \mathbb{Z}$. Therefore

$$a = kn + b = kn + qn + r = (k + q)n + r$$

so a has a remainder of r when divided by n, which is the same remainder as b.

Conversely, suppose $a = qn + r$ and $b = tn + r$ for some integers q, t and $0 \leq r < n$. Then $a - b = (q - t)n$ so $a \equiv b \pmod{n}$. ∎

As always, whenever we see an if and only if statement, we can now treat either side of the statement as the definition. The next proposition is not terribly difficult to prove, but it is vitally important. It tells us that the relation "congruent mod n" is an equivalence relation. See Appendix B for a crash course on equivalence relations.

Proposition 25. *Let n be a positive integer. Then for all $a, b, c \in \mathbb{Z}$ we have*

- $a \equiv a \pmod{n}$,

- *If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$, and*

- *If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.*

Proof. The proof is left as an exercse.   ∎

As explained in Appendix B, the equivalence classes partition the integers into sets. The equivalence classes here are called congruence classes mod n.

Definition. Let n be a positive integer. For $a \in \mathbb{Z}$, define the **congruence class of a mod n** as the set

$$[a] = \{b \in \mathbb{Z} \mid a \equiv b \pmod{n}\}.$$

Unwrapping the definitions we have $[a] = [b]$ if and only if $a \equiv b \pmod{n}$.

Example. When $n = 4$ we have exactly 4 congruence classes. They are

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$
$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$
$$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$$
$$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}.$$

It turns out that $[4] = [0]$, $[5] = [1]$ and so on (you should check these claims for your self!). Visualising the congruence classes on the number line we have
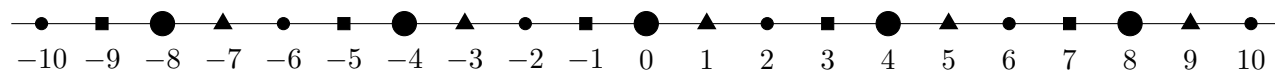
The ●s give the set $[0]$, the ▲s are $[1]$, the ●s are $[2]$ and the ■s are $[3]$. Notice that every integer belongs to exactly one congruence class, which is exactly what it means for the congruence classes to partition $\mathbb{Z}$.

4.2   The integers mod n

At the beginning of this section we played around with a clock with 7 things on it, and we called it $\mathbb{Z}_7$. We are now ready to formalise those ideas. It turns out that the set of congruence classes mod 7 are what should go on the clock with 7 things.

Definition. Let n be a positive integer. The set of all congruence classes mod n is called **the integers mod n** and is denoted $\mathbb{Z}_n$.

So, for example, $\mathbb{Z}_3 = \{[0], [1], [2]\}$. We want to do arithmetic on ZZ_n, in particular, we wish to add and multiply. Addition and multiplication on $\mathbb{Z}$ are operations that eat two elements of $\mathbb{Z}$ and spit out another element of $\mathbb{Z}$. Similarly, we want addition and multiplication on $\mathbb{Z}_n$ to eat two elements of $\mathbb{Z}_n$ (which are subsets of $\mathbb{Z}$) and spit out an element of $\mathbb{Z}_n$ (which is a subset of $\mathbb{Z}$).

Definition. Let n be a positive integer and define addition and multiplication on $\mathbb{Z}_n$ by

$$[a] + [b] = [a + b] \quad \text{and} \quad [a][b] = [ab].$$

So, for example, in $\mathbb{Z}_7$ we have $[3] + [5] = [8]$ but remember, $[8] = [1]$. So we could have just written $[3] + [5] = [1]$ which is what we want to do on the clock! Similarly, $[3][5] = [15] = [1]$.

There is something a little dodgy going on with the definition of addition and multiplication, which is that the definition appears to depend on what we call each equivalence class. Let's take $\mathbb{Z}_7$ again as an example. We have $[3] + [5] = [8]$, but what if we decided to write $[-4]$ or $[10]$ instead of $[3]$? Well, let's check: We have $[-4] + [5] = [1]$ and since $[1] = [8]$, all is well in the world and the result of the sum didn't depend on us writing $[3]$ as $[3]$ and not as $[-4]$. Similarly $[10] + [5] = [15] = [8]$. Phew! It would appear that no harm comes to us if we decide to call $[3]$ by any other of its infinitely many names. While a few examples are good, this should be checked!

Proposition 26. *Let n be a positive integer. If $[a] = [a']$ and $[b] = [b']$, then $[a] + [b] = [a'] + [b']$ and $[a][b] = [a'][b']$.*

Proof. If $[a] = [a']$ and $[b] = [b']$ then $n \mid a - a'$ and $n \mid b - b'$. Then $n \mid (a - a') + (b - b')$ so $n \mid (a + b) - (a' + b')$. Alas, $[a + b] = [a' + b']$. For multiplication we have $a = a' + kn$ and $b = b' + tn$ for some $k, t \in \mathbb{Z}$. Then $ab = (a' + kn)(b' + tn) = a'b' + n(kb' + ta' + nkt)$ so $[ab] = [a'b']$. ∎

Fantastic! No harm will come to us if in $\mathbb{Z}_9$ we decide to write $[2]$ as $[-7]$ or $[8]$ as $[-1]$.

Although no harm will come to you if you change the name of congruence classes when doing multiplication or addition (and since subtraction is just adding negatives, no harm comes while subtracting either), harm will come to you if you mess around with powers! For example, in $\mathbb{Z}_3$ $[2]^4 = [2][2][2][2] = [16] = [1]$ but $[2]^1 = [2]$. Powers are just an instruction for how many times you should multiply something by itself, which is not something that can be changed!

Exercise. Prove or disprove: Let n be a positive integer. Assume $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, and suppose $b, b' > 0$. Then $a^b \equiv a'^{b'} \pmod{n}$.

Now that we've checked the details to make sure addition and multiplication are well-defined in $\mathbb{Z}_n$, let's bask in our newfound glory.

Example. Let's compute the remainder of $(32)(33)$ when divided by 37. In $\mathbb{Z}_{37}$ we have $[(32)(33)] = [32][33] = [-5][-4] = [20]$. Therefore $(32)(33) \equiv 20 \pmod{37}$ so the remainder is 20.

Example. Let's compute $[3]^8$ in $\mathbb{Z}_{13}$. We have

$$[3]^2 = [9]$$
$$\Rightarrow [3]^3 = [3]^2[3] = [9][3] = [27] = [1]$$
$$\Rightarrow [3]^6 = ([3]^3)^2 = [1]$$
$$\Rightarrow [3]^8 = [3]^6[3]^2 = [1][9] = [9].$$

So, in particular, we know that 3^8 leaves a remainder of 9 when divided by 13.

Lecture 14 - 05/02

Example. Let's prove that for all $a \in \mathbb{Z}$, $a(a + 1)(a + 2)$ is divisible by 6. This is equivalent to showing $[a(a + 1)(a + 2)] = [a]([a] + [1])([a] + [2]) = [0]$ in $\mathbb{Z}_6$. So, we just have to check all the possible values of $[a]$ in $\mathbb{Z}_6$. Let's do it!

If $[a] = [0]$ we have $[0]([0] + [1])([0] + [2]) = [0]$.
If $[a] = [1]$ we have $[1]([1] + [1])([1] + [2]) = [1][2][3] = [6] = [0]$.
If $[a] = [2]$ we have $[2]([2] + [1])([2] + [2]) = [2][3][4] = [0][4] = [0]$.

28

If $[a] = [3]$ we have $[3]([3] + [1])([3] + [2]) = [3][4][5] = [0]$.
If $[a] = [4]$ we have $[4]([4] + [1])([4] + [2]) = [4][5][0] = [0]$.
Finally, if $[a] = [5] = [-1]$ we have $[-1]([-1] + [1])([-1] + [2]) = [-1][0][1] = [0]$.
Since $[a]([a] + [1])([a] + [2]) = [0]$ for all $[a] \in \mathbb{Z}_6$, $a(a + 1)(a + 2) \equiv 0 \pmod 6$ for all $a \in \mathbb{Z}$.

The fantastic thing about the previous example is that it turns checking infinitely many things (namely every integer, which is not possible) into a problem which needs you to check finitely many things! Although sometimes the finite number is large, given enough time and money you can always just do it.

Example. Let's show there is no integer solution to the equation $x^2 = 2736483623$, that is we want to show 2736483623 is not a perfect square.

We will approach this problem in $\mathbb{Z}_4$ (it will become clear as we progres as to why we chose 4). If there is an $x \in \mathbb{Z}$ so that $x^2 = 2736483623$, then it is true that $[x]^2 = [2736483623]$ in $\mathbb{Z}_4$ (in fact it's true in $\mathbb{Z}_n$ for any n). Let's take a look at the possible values of $[x]^2$.

If $[x] = [0]$ then $[x]^2 = [0]$. If $[x] = [\pm 1]$, then $[x]^2 = [1]$. If $[x] = [2]$ then $[x]^2] = [0]$. Therefore in $\mathbb{Z}_4$, $[x]^2$ is either $[0]$ or $[1]$.

However, $[2736483623] = [23] + [100][27364836] = [23] = [3]$ and $[3]$ is not one of the possible values of $\mathbb{Z}_4$ obtained by a perfect square. Therefore 2736483623 is not a perfect square.

It is not always clear which $\mathbb{Z}_n$ to investigate. Which one to go for is part of the fun, and you get better at it with experience. Often however, it's just trial and error!. You may often choose an n that's not helpful.

In the previous example, if we had looked at $\mathbb{Z}_{11}$ we would have $[2736483623] = [5]$ but $[4]^2 = [5]$. This tells us there is an $[x] \in \mathbb{Z}_{11}$ so that $[x]^2 = [2736483623]$, but we cannot conclude anything about whether or not there is an $x \in \mathbb{Z}$ so that $x^2 = 2736483623$.

The previous example was a special case of showing that a polynomial does not have any integer roots. Since polynomials are just made up of addition and multiplication, they are perfect equations to probe using modular arithmetic.

Proposition 27. *Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients, and let n be a positive integer. If $[b] = [c]$ in $\mathbb{Z}_n$, then $[a_n][b]^n + \cdots + [a_1][b] + [a_0] = [a_n][c]^n + \cdots + [a_1][c] + [a_0]$ in $\mathbb{Z}_n$.*

Proof. This is an exercise. ∎

Exercise. Show that the polynomial $x^5 - x^2 + x - 3$ has no integer roots.

Exercise. Is it true that if a polynomial with integer coefficients $f(x)$ has a root in $\mathbb{Z}_n$ for all primes n, then $f(x)$ has a root in $\mathbb{Z}$?

Sometimes when solving an equation in $\mathbb{Z}_n$, it's helpful to split it up into a few different equations.

Lemma 28. *Let m and n be positive coprime integers. Then $a \equiv b \mod mn$ if and only if $a \equiv b \pmod n$ and $a \equiv b \pmod m$.*

Proof. Suppose $a \equiv b \pmod{mn}$. Then $mn \mid a - b$ and since $m \mid mn$ and $n \mid mn$, $m \mid a - b$ and $n \mid a - b$. Conversely, suppose $a \equiv b \pmod m$ and $a \equiv b \pmod n$, so $mk = a - b$ and $nt = a - b$

for some $k, t \in \mathbb{Z}$. Since $\gcd(n, m) = 1$, there exist $u, v \in \mathbb{Z}$ so that $mu + nv = 1$. Multiplying both sides by $a - b$ gives

$$
\begin{aligned}
(a - b)mu + (a - b)nv &= a - b \\
\implies \quad (nt)mu + (mk)nv &= a - b \\
\implies \quad nm(tu + kv) &= a - b
\end{aligned}
$$

Thereore $a \equiv b \pmod{mn}$, completing the proof. $\blacksquare$

Exercise. Let $n = p_1^{e_1} \cdots p_k^{e_k}$ where the p_i are distinct primes and $e_i > 0$ for all i. Prove that $a \equiv b \pmod{n}$ if and only if $a \equiv b \pmod{p^{e_i}}$ for all i.

Of course, as is often the case with results that involve coprime integers and their product, there is a generalisation to all pairs of integers, but you have to replace their product with their least common multiple.

Exercise. Prove that for all positive integers m, n, $a \equiv b \pmod{\operatorname{lcm}(m, n)}$ if and only if $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$.

Let's use this to our advantage.

Exercise. Show that for all $a \in \mathbb{Z}$, $2 \mid a(a + 1)(a + 2)$ and $3 \mid a(a + 1)(a + 2)$. Conclude that for all $a \in \mathbb{Z}$, $6 \mid a(a + 1)(a + 2)$.

5 Inverses and Euler's Totient Function

In the previous section we built up the required machinery to treat $\mathbb{Z}_n$ as an object, like the integers, which comes with its own addition and multiplication. In mathematics, such an object is called a **ring**. We won't give the formal definition here, but a ring is a set endowed with addition and multiplication satisfying certain desirable properties.

A couple of the important properties are the existence of additive and multiplicative identities, which we usually call 0 and 1. The additive identity 0 has the property that $0 + a = a$ for all a in the ring. The multiplicative identity 1 has the property that $1a = a$ for all a in the ring.

Exercise. Let n be a positive integer.

- Which element of $\mathbb{Z}_n$ plays the role of 0 (the additive identity)?

- Which element of $\mathbb{Z}_n$ is $-[a]$? In a ring, $-x$ is the element with the property that $(-x) + x = 0$.

- Which element of $\mathbb{Z}_n$ plays the role of 1 (the multiplicative identity)?

- When element of $\mathbb{Z}_n$ is $[a]^{-1}$? In a ring x^{-1} is the element with the property that $x^{-1}x = 1$.

Lecture 15 - 10/02

Let's investigate the last part of this exercise, finding inverses in $\mathbb{Z}_n$.

5.1 Units in $\mathbb{Z}_n$

Consider $\mathbb{Z}_7$. What is the inverse of $[2]$? Now, it's not even clear that one exists. After all, in $\mathbb{Z}$ we know 0 doesn't have an inverse (in fact the only elements with inverses are ± 1). However, if $[2]^{-1}$ does exist, it has the property $[2][2]^{-1} = [1]$. A quick check through the elements of $\mathbb{Z}_7$ gives us $[2]^{-1} = [4]$ since $[2][4] = [1]$. Let's write down all the inverses in $\mathbb{Z}_7$.

x	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$	$[6]$
x^{-1}	$*$	$[1]$	$[4]$	$[5]$	$[2]$	$[3]$	$[6]$

So, just like the rational numbers $\mathbb{Q}$, every non-zero element has an inverse! Let's do another one, $\mathbb{Z}_{12}$.

x	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$	$[6]$	$[7]$	$[8]$	$[9]$	$[10]$	$[11]$
x^{-1}	$*$	$[1]$	$*$	$*$	$*$	$[5]$	$*$	$[7]$	$*$	$*$	$*$	$[11]$

Curious! Only some of the elements in $\mathbb{Z}_{12}$ have inverses. From these two examples you may be able to take a guess as to which elements in $\mathbb{Z}_n$ have inverses.

Definition. An element $x \in \mathbb{Z}_n$ is a **unit** if it has a multiplicative inverse. The set of all units in $\mathbb{Z}_n$ is called the **group of units mod** n and is denoted $\mathbb{Z}_n^*$.

So, for example, $\mathbb{Z}_{12}^* = \{[1], [5], [7], [11]\}$.

Theorem 29. *Let n be a positive number. An element $[a] \in \mathbb{Z}_n$ is a unit if and only if $\gcd(a, n) = 1$.*

Proof. If $\gcd(a, n) = 1$ there exist $x, y \in \mathbb{Z}$ so that $ax + ny = 1$, so $ax \equiv 1 \pmod{n}$. Therefore there exists $x \in \mathbb{Z}$ so that $[a][x] = [1]$ in $\mathbb{Z}_n$ and $[a] \in \mathbb{Z}_n^*$. Conversely, suppose $[a][x] = [1]$ in $\mathbb{Z}_n$ for some $x \in \mathbb{Z}$. Then $ax = 1 + ny$ for some $y \in \mathbb{Z}$, which rearranges to $ax - ny = 1$ Therefore $\gcd(a, n) = 1$, completing the proof. ∎

From this theorem we get the next quick consequence.

Corollary 30. *Let p be a prime. Then $[x] \in \mathbb{Z}_p^*$ if and only if $[x] \neq [0]$.*

Proof. A fun, for some definition of fun, exercise. ∎

This corollary tell us that $\mathbb{Z}_p$ acts a lot like $\mathbb{Q}$ or $\mathbb{R}$ in that every non-zero element is invertible. These are examples of fields.

Definition. We say $\mathbb{Z}_n$ is a **field** if every non-zero element is a unit.

Exercise. Prove that if $\mathbb{Z}_n$ is a field, then n is prime.

"This is all well and good," I hear you start, "but who cares?" Well, let me tell you! The existence of inverses allows us to do division (because division is just multplication by an inverse). In the real numbers $\mathbb{R}$ we were always taught "you can divide by anything except for 0. Dividing by 0 is bad!" The reason we were taught this is that every element of $\mathbb{R}$ has an inverse except for zero. This means we can divide by $a \neq 0$ by simply multiplying by a^{-1}.

Example. Suppose we want to solve the congruence $3x \equiv 4 \pmod{7}$. This is the same as solving $[3][x] = [4]$ in $\mathbb{Z}_7$. We can simply divide both sides by $[3]$, which really means multiplying both sides by $[3]^{-1} = [5]$. This gives $[5][3][x] = [5][4]$ implying $[x] = [20] = [6]$. So the solution to the original congruence is $x \equiv 6 \pmod{7}$.

5.2 Euler's Theorem and a not-big theorem of Fermat

We saw in Section 4.2 that modular arithmetic can help us deal with powers. More specifically, we easily computed that $[3]^8 = [9]$ in $\mathbb{Z}_{13}$.

Let's explore a little more. Suppose we needed to find $[3]^{100}$ in $\mathbb{Z}_{13}$. Let's write out some powers of $[3]$ in $\mathbb{Z}_{13}$.

n	x^n
1	$[3]$
2	$[9]$
3	$[1]$
4	$[3]$
5	$[9]$
6	$[1]$
7	$[3]$
8	$[9]$
9	$[1]$

Let's stop for a moment and stare at this table. It's repeating! Since a $[1]$ appears, we should absolutely expect this to happen. After all if $[3]^k = [1]$, then $[3]^{k+1} = [3]^k[3] = [3]$, taking us back to the start. In this case, we have that if $3 \mid k$, then $[3]^k = [1]$. Therefore

$$[3]^{100} = [3]^{99}[3] = [1][3] = [3]$$

in $\mathbb{Z}_{13}$. Let's see if this kind of repeating patterns happens more generally.

Here are power tables for the elements of $\mathbb{Z}_5$, $\mathbb{Z}_7$ and $\mathbb{Z}_9$. To make things easier to read, I will simply write $[a]$ as a.

$\mathbb{Z}_5$	0	1	2	3	4
x^1	0	1	2	3	4
x^2	0	1	4	4	1
x^3	0	1	3	2	4
x^4	0	1	1	1	1
x^5	0	1	2	3	4
x^6	0	1	4	4	1

$\mathbb{Z}_7$	0	1	2	3	4	5	6
x^1	0	1	2	3	4	5	6
x^2	0	1	4	2	2	4	1
x^3	0	1	1	6	1	6	6
x^4	0	1	2	4	4	2	1
x^5	0	1	4	5	2	3	6
x^6	0	1	1	1	1	1	1
x^7	0	1	2	3	4	5	6
x^8	0	1	4	2	2	4	1

$\mathbb{Z}_9$	0	1	2	3	4	5	6	7	8
x^1	0	1	2	3	4	5	6	7	8
x^2	0	1	4	0	7	7	0	4	1
x^3	0	1	8	0	1	8	0	1	8
x^4	0	1	7	0	4	4	0	7	1
x^5	0	1	5	0	7	2	0	4	8
x^6	0	1	1	0	1	1	0	1	1
x^7	0	1	2	0	4	5	0	7	8
x^8	0	1	4	0	7	7	0	4	1

There is a myriad of things to discover among these tables, and we will return to them at various points in the course. In fact they're so important I insist you do the following exercise.

Exercise (Super important!!). Create power tables for $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{13}$ and $\mathbb{Z}_{17}$. Stare at them all and look for patterns. Write down anything you notice and try to prove any conjectures you may have. Keep these tables safe and accessible for the rest of the course.

Lecture 16 - 02/10

Let's pay attention to the 4th powers in the first power table above, and the 6th powers in the other two. All those rows only consist of 0s and 1s. Even better, the 1s correspond exactly to the units. Let's rewrite these tables, but this time only for the units. Also, we know that once any column gets to a 1, it just repeats, so we'll stop if we see a row of 1s.

$\mathbb{Z}_5^*$	1	2	3	4
x^1	1	2	3	4
x^2	1	4	4	1
x^3	1	3	2	4
x^4	1	1	1	1

$\mathbb{Z}_7^*$	1	2	3	4	5	6
x^1	1	2	3	4	5	6
x^2	1	4	2	2	4	1
x^3	1	1	6	1	6	6
x^4	1	2	4	4	2	1
x^5	1	4	5	2	3	6
x^6	1	1	1	1	1	1

$\mathbb{Z}_9^*$	1	2	4	5	7	8
x^1	1	2	4	5	7	8
x^2	1	4	7	7	4	1
x^3	1	8	1	8	1	8
x^4	1	7	4	4	7	1
x^5	1	5	7	2	4	8
x^6	1	1	1	1	1	1

So, it would appear that if we only look at the units in $\mathbb{Z}_n$, some power of all of the units is 1. Even better, it appears that the special power is equal to the number of units! It would be a cruel cruel joke if this were just a coincidence.

Before we write down this conjecture and try to prove it, we need to introduce the totient function.

Definition. Define **Euler's totient function** (or **Euler's phi function**) to be the function defined by $\varphi(n) = |\mathbb{Z}_n^*|$ for any positive integer n.

Theorem 31 (Euler's Theorem). *Let n be a positive integer, and let $[a] \in \mathbb{Z}_n^*$. Then $[a]^{\varphi(n)} = [1]$ in $\mathbb{Z}_n$.*

Before we embark on the proof, we will need an interesting fact about the group of units. To demonstrate this fact, let's look at the multiplication table for $\mathbb{Z}_5^*$ (again, let's abuse notation and write $[a]$ as a):

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

There are two things worth noticing here. First, every entry in the table is again in $\mathbb{Z}_n^*$.

Exercise. Prove that if $[a], [b] \in \mathbb{Z}_n^*$, then $[a][b] \in \mathbb{Z}_n^*$.

The second thing is that every row and every column contains all the elements of $\mathbb{Z}_5^*$ exactly once! Or, more formally:

Lemma 32. *Let $[a] \in \mathbb{Z}_n^*$. Then the function $\psi : \mathbb{Z}_n^* \to \mathbb{Z}_n^*$ given by $\psi([b]) = [a][b]$ is a bijection.*

Proof. Note that since the product of two units is a unit, ψ is well-defined (that is $\psi([b]) \in \mathbb{Z}_n^*$ for every $[b] \in \mathbb{Z}_n^*$).

For injectivity, suppose $\psi([b]) = \psi([c])$, so $[a][b] = [a][c]$. Then since $[a]$ is a unit, multiplying both sides by $[a]^{-1}$ gives $[b] = [c]$ and ψ is injective. For surjectivity, let $[c] \in \mathbb{Z}_n^*$. Since $[a]^{-1}$ and $[c]$ are units, so is $[a]^{-1}[c]$. Then $\psi([a]^{-1}[c]) = [a][a]^{-1}[c] = [c]$, so ψ is surjective, and thus a bijection. ∎

We are now ready to prove Euler's theorem.

Proof of Euler's Theorem. Let $\mathbb{Z}_n^* = \{[u_1], \ldots, [u_{\varphi(n)}]\}$. By Lemma 32, for all $[a] \in \mathbb{Z}_n^*$ we have $\mathbb{Z}_n^* = \{[a][u_1], \ldots, [a][u_{\varphi(n)}]\}$.

Consider the product $[u_1][u_2] \cdots [u_{\varphi(n)}]$. Let $[a] \in \mathbb{Z}_n^*$. We have

$$[u_1][u_2] \cdots [u_{\varphi(n)}] = ([a][u_1])([a][u_2]) \cdots ([a][u_{\varphi(n)}]) = [a]^{\varphi(n)}[u_1][u_2] \cdots [u_{\varphi(n)}].$$

Since each $[u_i]$ is a unit, so is the product $[u_1][u_2]\cdots[u_{\varphi(n)}]$. So we can cancel the product from the equation above to get $[a]^{\varphi(n)} = [1]$. ∎

When n is a prime we recover Fermat's Little Theorem.

Corollary 33 (Fermat's Little Theorem)**.** *Let p be a prime and suppose $a \in \mathbb{Z}$ with $p \nmid a$. Then $a^{p-1} \equiv 1 \pmod{p}$.*

Example. We have $\mathbb{Z}_{15}^* = \{[1], [2], [4], [7], [8], [11], [13], [14]\}$ so $\varphi(15) = 8$. Therefore $15 \mid 7^{8088} - 1$. I'm sure it was keeping you up at night, but now you know that $7^{8088} - 1$ is indeed a multiple of 15.

Exercise. Let p and q be distinct primes, and let $a \in \mathbb{Z}$ be such that $p \nmid a$ and $q \nmid a$.

1. Prove that $a^{p^2 - p} \equiv 1 \pmod{p^2}$.

2. Prove that $a^{(p-1)(q-1)} \equiv 1 \pmod{pq}$.

5.3 Euler's Totient Function

As we saw in the previous section, being able to compute the totient function $\varphi(n)$ is useful! Let's investigate this function some more, and we will see that it shares lots of properties with lots of important functions in number theory (called multiplicative functions).

Let's begin with a small example.

Example. Let's work out $\varphi(49)$. So we're looking for all integers n coprime to 49 such that $0 \leq n < 49$. The prime factorisation of 49 is 7^2, so $\gcd(n, 49) = 1$ iff $7 \nmid n$. There are exactly 7 multiples of 7 from 0 to 48, so $\varphi(49) = 49 - 7 = 41$.

This kind of argument works well in general for powers of primes.

Proposition 34. *Let p be a prime and e a positive integer. Then $\varphi(p^e) = p^e - p^{e-1}$.*

Proof. The number of integers in $\{0, \ldots, p^e - 1\}$ coprime to p^e is exactly the number of integers in $\{0, \ldots, p^e - 1\}$ not divisible by p. There are p^{e-1} multiples of p in $\{0, \ldots, p^e - 1\}$ so $\varphi(p^e) = p^e - p^{e-1}$. ∎

In general, if we have a function defined on the positive integers and we know how it behaves for prime powers, and for products of coprime integers, then by looking at prime factorisations, we know how the function behaves for all positive integers. With that in mind, let's shift our focus to computing $\varphi(mn)$ where $\gcd(m, n) = 1$.

Let's look at the case $m = 9$ and $n = 5$. We want to count the number of integers in $\{0, \ldots, 44\}$ that are coprime to 45. Let's arrange these integers in an array as follows.

$$
\begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\
27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 \\
36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44
\end{array}
$$

34

The **boldface blue** entries represent the integers coprime to 45.

There are three things to notice in this array.

- Each column corresponds to a congruence class in $\mathbb{Z}_9$.

- In each column, every congruence class of $\mathbb{Z}_5$ appears exactly once. For example, in the column corresponding to $[1]$ in $\mathbb{Z}_9$, we have $1, 10, 19, 28, 37$. Intriguingly, $\mathbb{Z}_5 = \{[1], [10], [19], [28], [37]\}$.

- The **boldface blue** integers are precisely those that are coprime to both 5 and 9.

With these observations, let's figure out an expression for $\varphi(45)$ in terms of $\varphi(9)$ and $\varphi(5)$. Since the columns correspond to elements of $\mathbb{Z}_9$, our desired integers must appear in columns corresponding to elements of $\mathbb{Z}_9^*$ (since these are precisely the integers coprime to 9). There are $\varphi(9) = 6$ such columns. In each of these columns, since every element of $\mathbb{Z}_5$ is represented precisely once, there are exactly $\varphi(5) = 4$ integers which are coprime to 5. So there are $\varphi(9)$ columns with $\varphi(5)$ integers coprime to both 9 and 5 (and thus 45) in each column, giving $\varphi(45) = \varphi(9)\varphi(5) = 24$.

Let's prove that this kind of argument works in general.

Lemma 35. *Let $a, m, n \in \mathbb{Z}$. We have $\gcd(a, m) = 1$ and $\gcd(a, n) = 1$ if and only if $\gcd(a, mn) = 1$.*

Proof. Suppose $\gcd(a, mn) = 1$, so there exist $u, v \in \mathbb{Z}$ such that $au + mnv = 1$. Then $a(u) + m(nv) = 1$ and $a(u) + n(mv) = 1$ so $\gcd(a, m) = \gcd(a, n) = 1$. Conversely, suppose $\gcd(a, m) = \gcd(a, n) = 1$. Therefore there are $w, x, y, z \in \mathbb{Z}$ so that $aw + mx = 1$ and $ay + nz = 1$. Multiplying these equations gives

$$(aw + mx)(ay + nz) = 1$$
$$\implies a(awy + wnz + mxy) + mn(xz) = 1$$

and we can conclude $\gcd(a, mn) = 1$. ∎

Good! This lemma tells us that to count the integers coprime to mn, we need to count the integers that are coprime to m and n.

Now, suppose we have an $m \times n$ array as above with $\gcd(m, n) = 1$. We need that every column contains every element of $\mathbb{Z}_n$. The next lemma will be prove useful for this.

Lemma 36. *Let m and n be coprime positive integers. Let $c \in \mathbb{Z}$. Then the function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f([a]) = [a][m] + [c]$ is a bijection.*

Proof. Since $\gcd(m, n) = 1$, $[m]^{-1}$ exists. We will show that the function $g : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $g([a]) = [m]^{-1}([a] - [c])$ satisfies $gf([a]) = fg([a]) = [a]$ for all $[a] \in \mathbb{Z}_n$. We have

$$gf([a]) = g([a][m] + [c]) = [m]^{-1}(([a][m] + [c]) - [c]) = [a] + [m]^{-1}[c] - [m]^{-1}[c] = [a]$$

and

$$fg([a]) = f([m]^{-1}([a] - [c])) = [m]([m]^{-1}([a] - [c])) + [c] = [a] - [c] + [c] = [a].$$

Therefore f is a bijection. ∎

Now let's prove that $\varphi(mn) = \varphi(m)\varphi(n)$ if $\gcd(m, n) = 1$.

Proposition 37. *Let m and n be positive coprime integers. Then $\varphi(mn) = \varphi(m)\varphi(n)$.*

Proof. We want to count the number of integers in $\{0, 1, \ldots, mn - 1\}$ that are coprime to both m and n. We will write this set as

$$\{j + mi : 0 \leq j \leq m - 1, 0 \leq i \leq n - 1\}.$$

If we want to think of the array from the previous examples, the entry $j + mi$ is in the ith row and jth column, so the array has n rows and m columns.

Fix t such that $0 \leq t \leq m - 1$ and define the set $C_t = \{t + mi : 0 \leq i \leq n - 1\}$ (this is the column corresponding to the congruence class of $[t] \in \mathbb{Z}_m$).

Since $\gcd(m, j + mi) = \gcd(m, j)$, the integers that are coprime to m are precisely those in the columns C_t where $\gcd(m, t) = 1$. There are exactly $\varphi(m)$ such choices of t.

For a fixed t, we can write

$$C_t = \{t + m(0), t + m(1), t + m(2), \ldots, t + m(n - 1)\}.$$

Since $\gcd(m, n) = 1$, by Lemma 36 there is a bijection $f : \mathbb{Z}_n \to \mathbb{Z}_n$ such that $f([i]) = [t + m(i)]$. Therefore C_t contains exactly one representative of each congruence class in $\mathbb{Z}_n$, or equivalently, $\{[t + m(0)], [t + m(1)], [t + m(2)], \ldots, [t + m(n - 1)]\} = \mathbb{Z}_n$. We can conclude that each set C_t contains $\varphi(n)$ integers coprime to n.

There are $\varphi(m)$ values of t so that $\gcd(k, m) = 1$ if and only if $k \in C_t$. Also, each C_t contains exactly $\varphi(n)$ elements coprime to n. Therefore there are $\varphi(n)\varphi(m)$ integers coprime to mn, so $\varphi(n)\varphi(m) = \varphi(nm)$. $\blacksquare$

Lecture 18 (14/02)

Propositions 34 and 37 allow us to compute $\varphi(n)$ for any integer n.

Theorem 38. *Suppose a positive integer has prime factorisation $n = p_1^{e_1} \cdots p_k^{e_k}$ where the p_i are distinct prime factors and $e_i > 0$ for all i. Then*

$$\varphi(n) = \prod_{i=1}^{k} \left(p_i^{e_i} - p_i^{e_i - 1} \right) = \prod_{i=1}^{k} p_i^{e_i - 1}(p_i - 1) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i} \right).$$

Proof. Note that if p and q are distinct primes, then $\gcd(p^e, q^f) = 1$. With this in mind, the proof is left as an exercise (proceed by induction on k). $\blacksquare$

We may also write

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i} \right) = n \prod_{p \mid n} \left(1 - \frac{1}{p} \right)$$

which means the product is taken over all primes dividing n.

Example. We have $120 = 2^3 \cdot 3 \cdot 5$ so $\varphi(120) = 120(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5}) = 120(\frac{1}{2})(\frac{2}{3})(\frac{4}{5}) = 32$.

Exercise. Show there are integers n with $\varphi(n) = 2, 4, 6, 8, 10$, and 12, but not 14.

We finish our study of the totient function with an important property that will rear its head again as the course goes on.

The function $\varphi(n)$ counts the number of integers a in $\{0, \ldots, n - 1\}$ that have $\gcd(a, n) = 1$. Those of you with a keen eye will notice that in general, 1 is not the only possible value for $\gcd(a, n)$. Why don't we count the others? Great question. Let's count them for $n = 12$.

For 12, the possible values of $\gcd(a, 12)$ are precisely the positive divisors of 12. That is, $\gcd(a, 12) \in \{1, 2, 3, 4, 6, 12\}$ for all $a \in \mathbb{Z}$. Let's count how many integers a in $\{0, \ldots, 11\}$ have each of these greatest common divisors with 12.

$\gcd(a, 12)$	a	How many a?
1	1,5,7,11	4
2	2,10	2
3	3,9	2
4	4,8	2
6	6	1
12	0	1

It's not so clear that there's anything interesting to say here, but let's add another column, this time writing down $\frac{a}{\gcd(a,12)}$.

$\gcd(a, 12)$	a	How many a?	$\frac{a}{\gcd(a,12)}$
1	1,5,7,11	4	1,5,7,11
2	2,10	2	1,5
3	3,9	2	1,3
4	4,8	2	1,2
6	6	1	1
12	0	1	0

In the last column, the first row is a set of representatives for the elements of $\mathbb{Z}_{12}^*$. The second row is precisely the set of integers in $\{0, \ldots, 5\}$ that are coprime to 6. The third row is eactly those integers we count to compute $\varphi(4)$. The pattern continues! So, in the column that is counting how many a there are, we have $\varphi(12), \varphi(6), \varphi(4), \varphi(3), \varphi(2)$, and $\varphi(1)$ respectively. Since every number in $\{0, \ldots, 11\}$ falls into one of these rows, we have

$$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) = 12.$$

Coincidence? No way!

Lemma 39. *Let n be a positive integer and d a positive divisor of n. Let*

$$S_d = \left\{ a \in \{0, \ldots, n-1\} : \gcd(a, n) = \frac{n}{d} \right\}.$$

Then $|S_d| = \varphi(d)$.

Proof. Let $T_d = \{b \in \{0, \ldots, d-1\} : \gcd(b, d) = 1\}$ and note $\varphi(d) = |T_d|$. So, we want to show that $|S_d| = |T_d|$. Let $f : S_d \to T_d$ be given by $f(a) = \frac{ad}{n}$. It suffices to show f is a bijection.

To see f is well-defined, note that since $\gcd(a, n) = \frac{n}{d}$, $\frac{n}{d} \mid a$ so $\frac{ad}{n}$ is an integer. Furthermore, since $0 \le a < n$ and d and n are positive, $0 \le f(a) < \frac{nd}{n} = d$. Therefore $f(a) \in \{0, \ldots, d-1\}$. Finally, we need to check that $\gcd(f(a), d) = 1$. Since $\gcd(a, n) = \frac{n}{d}$, by Bezout's identity there exist $u, v \in \mathbb{Z}$ such that

$$au + nv = \frac{n}{d}$$
$$\implies \frac{ad}{n}u + \frac{nd}{n}v = \frac{n}{d}\frac{d}{n}$$
$$\implies f(a)u + dv = 1$$

so $\gcd(f(a), d) = 1$ and f is a well-defined function.

For injectivity, suppose $f(a) = f(a')$, so $\frac{ad}{n} = \frac{a'd}{n}$. Since $d \neq 0$ and $n \neq 0$, this implies $a = a'$ and f is injective. For surjectivity, let $b \in T_d$, so $\gcd(b, d) = 1$. Then by Bezout's identity there exist $u, v \in \mathbb{Z}$ so that

$$bu + dv = 1$$
$$\implies \left(\frac{n}{d}b\right) u + nv = \frac{n}{d}$$

so $\gcd(\frac{n}{d}b, n) \mid \frac{n}{d}$. Since $\frac{n}{d} \mid \frac{n}{d}b$ and $\frac{n}{d} \mid n$, we have $\frac{n}{d} \mid \gcd(\frac{n}{d}b, n)$. Since all integers involved here are positive, we can conclude $\gcd(\frac{n}{d}b, n) = \frac{n}{d}$. Therefore $\frac{n}{d}b \in S_d$ and $f(\frac{n}{d}b) = b$ so f is surjective. ∎

Lecture 19 (25/02)

Theorem 40. *Let n be a positive integer. Then*

$$\sum_{d \mid n} \varphi(d) = n$$

where $\sum_{d \mid n}$ denotes the sum over all positive divisors d of n.

Proof. The sets $\{S_d : d \mid n\}$ partition the set $\{0, \ldots, n-1\}$ (that is, every element of the set $\{0, \ldots, n-1\}$ belongs to exactly one of the sets S_d). In particular, $a \in S_{\frac{n}{\gcd(a,n)}}$. Let $d_1, \ldots, d_k$ be the positive divisors of n. Then $n = |S_{d_1}| + |S_{d_2}| + \cdots + |S_{d_k}|$ and by Lemma 39 we have $n = \sum_{d \mid n} \varphi(d)$. ∎

6 Multiplicative functions

You may have noticed by now that in number theory, we find ourselves counting things. A lot. For example, $\varphi(n)$ counts the number of congruence classes in $\mathbb{Z}_n^*$. We may wish to know how many squares there are for $\mathbb{Z}_n^*$. Once we have the prime factorisation of a number, we can easily count the number of divisors. Or perhaps, we wish to know how many squares there are in $\mathbb{Z}_n$. Or cubes!

Before we make a definition, let's see what's interesting about some of these functions. Consider Euler's phi function $\varphi(n) = |\mathbb{Z}_n^*|$. We know that if $\gcd(n, m) = 1$, $\varphi(nm) = \varphi(n)\varphi(m)$. This makes things awfully convenient if we want to compute $\varphi(n)$ in general. This property pops up over and over when counting things to do with the integers.

Here are some examples of multiplicative functions before we give the general definition.

Example. Euler's phi function, $\varphi(n) = |\mathbb{Z}_n^*|$ is an example of a multiplicative function. We know that when $\gcd(n, m) = 1$, $\varphi(nm) = \varphi(n)\varphi(m)$.

Example. The function $\tau(n)$ is defined to be the number of positive divisors of n. So, $\tau(12) = 6$ since the positive divisors of 12 are $1, 2, 3, 4, 6$, and 12.

The function $\sigma(n)$ is defined to be the sum of the positive divisors of n. So, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

Both τ and σ are called **divisor functions**, and we can write them

$$\tau(n) = \sum_{d \mid n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d \mid n} d.$$

We will see a little later on that whenever $\gcd(n, m) = 1$, $\tau(nm) = \tau(n)\tau(m)$ and $\sigma(nm) = \sigma(n)\sigma(m)$, although you can prove these properties directly now if you'd like!

Exercise. Prove that if $\gcd(n, m) = 1$, $\tau(nm) = \tau(n)\tau(m)$ and $\sigma(nm) = \sigma(n)\sigma(m)$.

All of these functions count something that depends on a positive integer n. So they are functions that eat positive integers, and generally spit out integers, although the following definition allows the functions to take value in the complex numbers.

Definition. An **arithmetic function** is a function $f : \mathbb{Z}_{>0} \to \mathbb{C}$. A **multiplicative function** is an arithmetic function f satisfying $f(nm) = f(n)f(m)$ whenever $\gcd(n, m) = 1$.

For our first result about multiplicative functions, for a positive integer n, let $D_n = \{a \in \mathbb{Z}_{>0} : a \mid n\}$, that is, the set of positive divisors of n.

Lemma 41. *Let n and m be positive coprime integers. Then the function $h : D_n \times D_m \to D_{nm}$ given by $h(a, b) = ab$ is a bijection.*

Proof. This proof is an exercise. ∎

We now have a sweet little lemma that helps us easily prove some functions are indeed multiplicative.

Lemma 42. *Let g be a multiplicative function and $f(n) = \sum_{d \mid n} g(d)$ for all n. Then f is a multiplicative function.*

Proof. Let n and m be positive coprime integers. Then $f(nm) = \sum_{d \mid nm} g(d)$. However, by the previous lemma,

$$f(nm) = \sum_{a \mid n} \sum_{b \mid m} g(ab) = \sum_{a \mid n} \sum_{b \mid m} g(a)g(b)$$

since if $a \mid n$ and $b \mid m$, $\gcd(a, b) = 1$. We can rewrite the latest expression as

$$\sum_{a \mid n} \sum_{b \mid m} g(a)g(b) = \left(\sum_{a \mid n} g(a) \right) \left(\sum_{b \mid m} g(b) \right)$$

so $f(nm) = f(n)f(m)$ as required. ∎

To apply this lemma, let's introduce the following multiplicative functions.

Definition. Define multiplicative functions $u(n) = 1$ and $N(n) = n$ for all $n \in \mathbb{Z}_{>0}$. The function u is sometimes referred to as the **unit function**.

Note that both N and u are multiplicative.

Theorem 43. *The divisor functions τ and σ are multiplicative.*

Proof. We have $\tau(n) = \sum_{d \mid n} 1 = \sum_{d \mid n} u(d)$ and $\sigma(n) = \sum_{d \mid n} N(d)$. Therefore τ and σ are multiplicative by Lemma 42. ∎

Exercise. Show that for every integer $k \geq 0$, the function $\sigma_k(n) = \sum_{d \mid n} d^k$ is multiplicative. Note that $\sigma_0 = \tau$ and $\sigma_1 = \sigma$.

Example. Let's write down a formula for $\tau(n)$ and $\sigma(n)$ in terms of the prime factorisation of n.
Let $n = p_1^{e_1} \cdots p_k^{e_k}$. Since τ and σ are multiplicative, we have

$$\tau(n) = \prod_{i=1}^{k} \tau(p_i^{e_i}) \quad \text{and} \quad \sigma(n) = \prod_{i=1}^{k} \sigma(p_i^{e_i}).$$

So, we just need to work out how τ and σ behave on prime powers.
The divisors of p^e are $1, p, p^2, \ldots, p^e$. Therefore $\tau(p^e) = e + 1$ and

$$\sigma(p^e) = 1 + p + p^2 + \cdots + p^e = \frac{p^{e+1} - 1}{p - 1}$$

since $\sigma(p^e)$ is just a geometric series. Therefore

$$\tau(n) = \prod_{i=1}^{k}(e_i + 1) \quad \text{and} \quad \sigma(n) = \prod_{i=1}^{k} \left(\frac{p_i^{e_i+1} - 1}{p_i - 1} \right).$$

6.1 Möbius inversion

Let's gather up a few similar looking identities from the previous sections.

$$N(n) = \sum_{d|n} \varphi(d)$$

$$\tau(n) = \sum_{d|n} u(d)$$

$$\sigma(n) = \sum_{d|n} N(d).$$

Equations like this, relate one multiplicative function with another. It turns out, there is a neat little way to invert these relations, and it relies on a surprising little function called the Möbius function.

Definition. Define the **Möbius function** $\mu : \mathbb{Z}_{>0} \to \mathbb{C}$ as the multiplicative function defined by

$$\mu(p^e) = \begin{cases} -1 & \text{if } e = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where p is a prime and $e \in \mathbb{Z}_{>0}$.

So, for example, $\mu(15) = \mu(5)\mu(3) = (-1)(-1) = 1$, and $\mu(12) = \mu(2^2)\mu(3) = 0$.

Exercise. Prove that $\mu(1) = 1$.

Let's see what we can build out of this function. We'll investigate the multiplicative function $I(n) = \sum_{d|n} \mu(d)$.

When $n = 1$ we have $I(1) = \mu(1) = 1$. We know I is multiplicative, so let's figure out what it does to prime powers. We have

$$I(p^e) = \mu(1) + \mu(p) + \mu(p^2) + \cdots + \mu(p^e) = \mu(1) + \mu(p) = 0.$$

Therefore we immediately conclude the following for $n \in \mathbb{Z}_{>0}$:

$$I(n) = \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We'll see little later why we call this function I (for identity). The Möbius function seems a little ad-hoc, but it turns out to play a vital role in the study of multiplicative functions.

Theorem 44 (Möbius inversion formula)**.** *Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be an arbitrary function, and define $F(n) = \sum_{d|n} f(d)$ for all $n \in \mathbb{Z}_{>0}$. Then*

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Proof. We have

$$\begin{aligned}
\sum_{d|n} \mu(d) F(\frac{n}{d}) &= \sum_{d|n} \mu(d) \sum_{d'|\frac{n}{d}} f(d') \\
&= \sum_{d'|n} f(d') \sum_{d|\frac{n}{d'}} \mu(d) \\
&= \sum_{d'|n} f(d') I(\frac{n}{d'}) \\
&= f(n)
\end{aligned}$$

completing the proof. ∎

This proof is rather opaque, so it's a good exercise to go through it line by line, justifying each step and trying to build some intuition.

Exercise. Justify each step in the previous proof.

The proof isn't terribly important, but the result does yield a very useful result.

Corollary 45. *Let f and F be arithmetic functions so that $F(n) = \sum_{d|n} f(d)$ for all $n \in \mathbb{Z}_{>0}$. Then f is multiplicative if and only if F is multiplicative.*

Proof. We have already proved that if f is multiplicative, then F is multiplicative in Lemma 42. For the converse, by the Möbius inversion formula we have

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Let m, n be positive coprime integers. Then

$$f(mn) = \sum_{d|mn} \mu(d) F\left(\frac{mn}{d}\right)$$

$$= \sum_{a|m} \sum_{b|n} \mu(ab) F\left(\frac{mn}{ab}\right)$$

$$= \sum_{a|m} \sum_{b|n} \mu(a)\mu(b) F\left(\frac{m}{a}\right) F\left(\frac{n}{b}\right)$$

$$= \left(\sum_{a|m} \mu(a) F\left(\frac{m}{a}\right)\right) \left(\sum_{b|n} \mu(b) F\left(\frac{n}{b}\right)\right)$$

$$= f(m) f(n)$$

completing the proof. ∎

Since N is a multiplicative function, this gives us a new (although secretly the same) proof that the totient function is multiplicative!

Lecture 21 - 28/02

Example. Let's apply the Möbius inversion formula to the equation $N(n) = \sum_{d|n} \varphi(d)$. We have

$$\varphi(n) = \sum_{d|n} \mu(d) N\left(\frac{n}{d}\right) = \sum_{d|n} \frac{n\mu(d)}{d} = n \sum_{d|n} \frac{\mu(d)}{d}.$$

Exercise. Apply the Möbius inversion formula to the following equations. What comes out?

$$I(n) = \sum_{d|n} \mu(d)$$

$$\tau(n) = \sum_{d|n} u(n)$$

$$\sigma(n) = \sum_{d|n} N(n).$$

7   The group of units

We saw earlier that the power tables lead us to Euler's Theorem and Fermat's Little theorem. Let's squeeze some more juice out of them. Here are some power tables for $\mathbb{Z}_n^*$ from earlier, as well as one for $n = 12$.

$\mathbb{Z}_7^*$	1	2	3	4	5	6
x^1	1	2	3	4	5	6
x^2	1	4	2	2	4	1
x^3	1	1	6	1	6	6
x^4	1	2	4	4	2	1
x^5	1	4	5	2	3	6
x^6	1	1	1	1	1	1

$\mathbb{Z}_5^*$	1	2	3	4
x^1	1	2	3	4
x^2	1	4	4	1
x^3	1	3	2	4
x^4	1	1	1	1

$\mathbb{Z}_9^*$	1	2	4	5	7	8
x^1	1	2	4	5	7	8
x^2	1	4	7	7	4	1
x^3	1	8	1	8	1	8
x^4	1	7	4	4	7	1
x^5	1	5	7	2	4	8
x^6	1	1	1	1	1	1

$\mathbb{Z}_{12}^*$	1	5	7	11
x^1	1	5	7	11
x^2	1	1	1	1
x^3	1	5	7	11
x^4	1	1	1	1

Let's pay even closer attention to where the 1s are. In the first three tables, the first row where all the entries are 1s is in the $x^{\varphi(n)}$ row. However, although the $x^{\varphi(12)}$ row is all 1s, it's not the first place it happens in $\mathbb{Z}_{12}^*$ (it also happens in the x^2 row).

In fact, there are special columns in the first three tables where the first 1 appears where Euler's theorem tells us there must be a 1, and every other unit appears in that column before then! For example, look at the column [3] in $\mathbb{Z}_7^*$. The first power that is [1] is $[3]^6 = [3]^{\varphi(7)}$, and every unit appears in that column. In other words, every unit is a power of [3] in $\mathbb{Z}_7$. There is no special column like that in $\mathbb{Z}_{12}^*$.

There is another thing worth noticing, which is apparent in the tables for $\mathbb{Z}_7^*$ and $\mathbb{Z}_9^*$. Besides in the first column, the 1s only appear in the rows corresponding to divisors of $\varphi(n)$. In particular, no entry in the x^1 or $x^{\varphi(n)-1}$ rows (besides in the first column) are 1. This phenomenon is perhaps not too surprising: If $[a]^k = 1$ for some positive k, and this is the first positive k where it happens, then all the 1s in the column corresponding to $[a]$ should appear in multiplies of k. If $\varphi(n)$ is not a multiple of k, then $[a]^{\varphi(n)} \neq [1]$, which would contradict Euler's theorem.

But enough handwaving, let's formalise these observations.

Definition. Let n be a positive integer and $[a] \in \mathbb{Z}_n^*$. The **order of** $[a]$ **in** $\mathbb{Z}_n^*$ (sometimes denoted ord($[a]$) or $|[a]|$) is the smallest positive integer k such that $[a]^k = [1]$ in $\mathbb{Z}_n$.

So, in $\mathbb{Z}_7^*$ for example, ord($[2]$) = 3, ord($[3]$) = 6 and ord($[6]$) = 2. It is always the case that ord($[1]$) = 1 in $\mathbb{Z}_n^*$.

Exercise. Let n be a positive integer and $[a] \in \mathbb{Z}_n^*$. Prove that if $[a]^p = [1]$ for some prime p, and $[a] \neq [1]$ in $\mathbb{Z}_n$, then ord($[a]$) = p.

Let's first prove that the order of an element in $\mathbb{Z}_n^*$ must divide $\varphi(n)$.

Proposition 46. *Let $[a] \in \mathbb{Z}_n^*$. If $[a]^k = [1]$ for some integer k, then* ord($[a]$) $\mid k$.

Proof. This is left as an exercise (Question 3 on Assignment 3). ∎

Exercise. Use Proposition 46 to prove that for $n > 2$, $\varphi(n)$ is even.

7.1 Mersenne Numbers

Let's see a couple of applications of this new point of view. First up, a slick proof that there are infinitely many primes (the first such proof in these notes).

Theorem 47. *There are infinitely many primes.*

Proof. Towards a contradiction, let p be the largest prime and let q be a prime divisor of $2^p - 1$. Then $q \mid 2^p - 1$ so $[2]^p = [1]$ in $\mathbb{Z}_q$. Since q is prime, $[2] \in \mathbb{Z}_q^*$. Since p is prime, ord($[2]$) = p and so $p \mid q - 1$ by Proposition 46. Therefore $p \leq q - 1$ so $p < q$, a contradiction. ∎

Numbers of the form $2^p - 1$ turn out to be important in the search for primes, and they are called Mersenne numbers. The first few Mersenne numbers are

$$3, 7, 31, 127$$

which are all prime. However, $2^{11} - 1 = 2047 = 23 \cdot 89$ so it's not prime. It is unknown whether or not there are infinitely many Mersenne numbers that are prime (called Mersenne primes), but it is currently the best way we know for finding large primes. In fact, there is a collaborate project

called the Great Internet Mersenne Prime Search (GIMPS). As of October 2024, GIMPS has found 18 Mersenne primes, 16 of which were the largest at the time of their discovery. which has found the 15 largest primes known. The largest prime currently known is the Mersenne prime

$$2^{136,279,841} - 1$$

which has $41,024,320$ digits. The prime was found on October 12, 2024.

While it is unknown whether or not there are infinitely many Mersenne primes, we can prove that any two Mersenne numbers are coprime.

Proposition 48. *Let m and n be positive coprime integers. Then $2^m - 1$ and $2^n - 1$ are coprime.*

Proof. Let $g = \gcd(2^m - 1, 2^n - 1)$, and note that since both $2^m - 1$ and $2^n - 1$ are odd, g is odd. Therefore $[2] \in \mathbb{Z}_g^*$. Since $g \mid 2^m - 1$ we have $[2]^m = [1]$, and similarly $[2]^n = [1]$, in $\mathbb{Z}_g$. Let $x, y \in \mathbb{Z}$ be such that $mx + ny = 1$. Then

$$[2]^1 = [2]^{mx+ny} = ([2]^m)^x([2]^n)^y = [1]$$

in $\mathbb{Z}_g$. Therefore $g \mid (2 - 1)$ so $g = 1$. ∎

Lecture 22 - 03/03

7.2 Primitive Roots

As we saw in the power tables above, $\mathbb{Z}_5^*$, $\mathbb{Z}_7^*$, and $\mathbb{Z}_9^*$ all have a special column which contains every unit. This is not the case for $\mathbb{Z}_{12}$. Having this property is desirable. Let's look at $\mathbb{Z}_7^*$ for example. The element $[3]$ gives rise to one of these columns, and we have $\mathbb{Z}_7^* = \{[3]^0, [3]^1, [3]^2, [3]^3, [3]^4, [3]^5\}$. So the powers of $[3]$ generate all of $\mathbb{Z}_7^*$. The existence of such elements will allow us to prove some powerful results.

Definition. An element $[a] \in \mathbb{Z}_n^*$ such that $\mathrm{ord}([a]) = \varphi(n)$ is called a **primitive root** mod n (or a **generator** of $\mathbb{Z}_n^*$).

Example. The primitive roots mod 7 are $[3]$ and $[5]$ (this can be verified by looking at the power table.

The element $[2] \in \mathbb{Z}_{11}^*$ is a generator since $[2]^2 \neq [1]$ and $[2]^5 = [-1] \neq [1]$ in $\mathbb{Z}_{11}$ (why is it enough to just check the second and fifth powers here?).

Exercise. Prove that $[a]$ is a generator in $\mathbb{Z}_n^*$ if and only if $[a]^{\varphi(n)/p} \neq [1]$ in $\mathbb{Z}_n$ for every prime p that divides $\varphi(n)$.

Exercise. Find all (if any) generators of $\mathbb{Z}_{19}^*$. How many are there?

Exercise. Suppose $[a]$ is a generator of $\mathbb{Z}_n^*$. How many generators are there?

You may ask why we use the term "generator" (or "primitive root" for that matter, but let's just focus on "generator" for now). It's because a generator generates every other element of $\mathbb{Z}_n^*$! More precisely:

Exercise. Suppose $[a]$ is a generator of $\mathbb{Z}_n^*$. Prove that for all $[b] \in \mathbb{Z}_n^*$, there exists a positive integer k so that $[a]^k = [b]$.

If a generator for $\mathbb{Z}_n^*$ exists, then we can deduce a lot about how powers of elements behave in $\mathbb{Z}_n^*$.

Example. Let's find all elements $x \in \mathbb{Z}_{19}$ so that $x^3 = [1]$.

First we'll show that $[2]$ is a generator of $\mathbb{Z}_{19}$. We have

$$[2]^2 = [4]$$
$$[2]^4 = [-3]$$
$$[2]^6 = [7]$$
$$[2]^8 = [9]$$
$$[2]^9 = [-1]$$

and since $[2]^6 \neq [1]$ and $[2]^9 \neq [1]$, $[2]$ is a generator of $\mathbb{Z}_{19}$.

Now, $[0]$ is not a solution to $x^3 = [1]$, so we only need to consider the units in $\mathbb{Z}_{19}$. Since $[2]$ is a generator of $\mathbb{Z}_{19}^*$, we have $\mathbb{Z}_{19}^* = \{[2]^k : 0 \leq k < 18\}$.

Now, suppose $[2]^k$ satisfies $[1] = ([2]^k)^3 = [2]^{3k}$. Since $\text{ord}([2]) = 18$, we know $18 \mid 3k$. It's a quick exercise to prove that $18 \mid 3k$ if and only if $6 \mid k$, so all the solutions are given by $[2]^0, [2]^6$, and $[2]^{12}$. We can compute these powers to be $[1]$, $[7]$, and $[11]$.

Of course, we could have just checked all 19 possibilities, but this is more fun!

Knowing about primitive roots helps us investigate solutions to polynomials, as in the previous example. Our next goal is to prove that generators exist for $\mathbb{Z}_p^*$ for all primes p. In order to do that, we need to investigate solutions to polynomials in $\mathbb{Z}_p$!

7.3 Polynomials in $\mathbb{Z}_p$

Just about everything we cover in this section works for polynomials over a field $\mathbb{F}$. Recall a **field** is a ring (a set with addition and multiplication) where every non-zero element has an inverse. For the sake of this course, the examples of fields to keep in mind for this section are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the rational numbers $\mathbb{Q}$, and of course, $\mathbb{Z}_p$ when p is prime. So whenever you see $\mathbb{F}$, you will be just fine replacing it with $\mathbb{Z}_p$.

We start with an extremely useful fact about fields.

Proposition 49. *For all elements $a, b \in \mathbb{F}$, if $ab = 0$ then $a = 0$ or $b = 0$.*

Proof. Suppose $ab = 0$ and $a \neq 0$. Then a is a unit so $ab = 0$ implies $a^{-1}ab = a^{-1} \cdot 0$ and we conclude $b = 0$. ∎

Exercise. Show that the above proposition doesn't hold in $\mathbb{Z}_n$ whenever n is composite. Show that the above proposition does hold in $\mathbb{Z}$. Therefore being a field is sufficient for the proposition to hold, but not necessary!

Now, onto polynomials!

Lecture 23 - 05/03

Definition. A polynomial in the variable x with coefficients in $\mathbb{F}$ is an expression of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a non-negative integer, $a_i \in \mathbb{F}$ for all i, and $a_n \neq 0$. We also say $f(x) = 0$ is a polynomial.

The **degree** of $f(x) = a_n x^n + \cdots + a_0$ is n, denoted $\deg(f(x)) = n$. The degree of the zero polynomial is undefined.

The set of all polynomials in x over $\mathbb{F}$ is denoted $\mathbb{F}[x]$. When the variable is clear from context, or unimportant, we may denote $f(x)$ simply by f.

Example. $3x^2 - 2x \in \mathbb{R}[x]$ is a degree 2 polynomial with real coeffients. $[3]x^2 - [2]x \in \mathbb{Z}_5[x]$ is a degree 2 polynomial with coefficients in $\mathbb{Z}_5$. We also have $[2] \in \mathbb{Z}_5$, which is a degree-0 polynomial.

The set of polynomials $\mathbb{F}[x]$, comes with multiplication and addition, just like $\mathbb{Z}_n$ is a set with multiplication and addition. This turns $\mathbb{F}[x]$ into a ring. The multiplication and addition works as you're used to, and depends on the addition and multiplication in $\mathbb{F}$.

Example. Let $f, g \in \mathbb{Z}_5[x]$ be $f = x^3 + [2]x - [1]$ and $g = [3]x^2 + [2]x + [1]$. Then

$$f + g = x^3 + [3]x^2 + [4]x$$

and

$$
\begin{aligned}
fg &= (x^2 + [2]x - [1])([3]x^3 + [2]x + [1]) \\
&= [3]x^6 + [2]x^3 + x^2 + [6]x^4 + [4]x^2 + [2]x + [-1]x^3 + [-2]x + [-1] \\
&= [3]x^6 + [6]x^4 + [1]x^3 + [5]x^2 + [0]x + [0] \\
&= [3]x^6 + x^4 + x^3.
\end{aligned}
$$

Of course, when we write x^3 we really mean $[1]x^3$, and we simply leave out terms of the form $[0]x^k$.

Exercise. Write down all the units in $\mathbb{Z}_{11}[x]$.

Up until this point there we haven't seemed to need $\mathbb{F}$ to be a field. In fact, we can define $\mathbb{Z}[x]$ and $\mathbb{Z}_n[x]$ as we have already, and these too are rings. However, for what is about to follow, it's important that $\mathbb{F}$ is a field.

Proposition 50. *Let $f, g \in \mathbb{F}[x]$ with $f \neq 0$ and $g \neq 0$. Then $\deg(fg) = \deg(f) + \deg(g)$.*

Proof. Let $f = a_n x^n + \cdots + a_0$ and $g = b_m x^m + \cdots + b_0$, with $a_n \neq 0$ and $b_m \neq 0$. Then $fg = a_n b_m x^{n+m} + h$ for some $h \in \mathbb{F}[x]$ with $\deg(h) < n + m$. We have $a_n b_m \neq 0$ since $a_n \neq 0$ and $b_m \neq 0$, implying $\deg(fg) = n + m$. ∎

Exercise. Suppose f and g are non-zero polynomials in $\mathbb{F}[x]$. Prove that if $f + g \neq 0$, then $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

Just like in $\mathbb{Z}$, we have a division algorithm in $\mathbb{F}[x]$, which is amazingly useful.

Theorem 51 (Division algorithm for polynomials)**.** *Let $f, g \in \mathbb{F}[x]$ with $g \neq 0$. Then there exist unique polynomials $q, r \in \mathbb{F}[x]$ so that $f = qg + r$, where $\deg(r) < \deg(g)$ or $r = 0$.*

Proof. We begin with existence, for which we will induct on the degree of f. For the base case, suppose $f = 0$ or $\deg(f) < \deg(g)$. Then taking $q = 0$ and $r = f$ suffices. Suppose now that $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ with $n \geq m$. Assume that for all polynomials $h \in \mathbb{F}[x]$ with $\deg(h) < n$, there exist polynomials $q, r \in \mathbb{F}[x]$ satisfying $h = qg + r$ and $\deg(r) < \deg(g)$ or $r = 0$.

Consider the polynomial
$$f'(x) = f(x) - \frac{a_n}{b_m}x^{n-m}g(x)$$

which is such that $\deg(f') < n$. Therefore, by the inductive assumption, there exist polynomials $q', r \in \mathbb{F}[x]$ so that
$$f(x) - \frac{a_n}{b_m}x^{n-m}g(x) = q'(x)g(x) + r(x)$$

where $\deg(r) < \deg(g)$ or $r = 0$. If we let $q(x) = q'(x) + \frac{a_n}{b_m}x^{n-m}$, we have $f(x) = q(x)g(x) + r(x)$, completing the existence part of the proof.

For uniqueness, suppose $f = qg + r = q'g + r'$ where $\deg(r) < \deg(g)$ or $r = 0$, and $\deg(r') < \deg(g)$ or $r' = 0$. Then $g(q - q') = r' - r$. If $r = r'$, then since $g \neq 0$, we have $q = q'$ and we're done. So suppose $r \neq r'$. Then $\deg(r - r') < \deg(g)$. Since $r - r' \neq 0$, we cannot have $q = q'$. Then $\deg(g(q - q')) \geq \deg(g)$, contradicting the fact that $\deg(r - r') < \deg(g)$. Alas, we are forced to conclude $r = r'$ and $q = q'$, completing the proof. ∎

Exercise. Show that the division algorithm fails in $\mathbb{Z}[x]$. *Hint: consider $g = 2x$.*

Polynomials themselves aren't functions, they are simply elements of a ring (so things you can add and multiply). However, polynomials in $\mathbb{F}[x]$ do indeed define functions from $\mathbb{F}$ to $\mathbb{F}$.

Definition. Let $f = a_nx^n + \cdots + a_0 \in \mathbb{F}[x]$. Define the function $f : \mathbb{F} \to \mathbb{F}$ by $f(b) = a_nb^n + \cdots + a_1b + a_0$.

For example, the polynomial $f(x) = x^2 \in \mathbb{R}[x]$ defines a function $f : \mathbb{R} \to \mathbb{R}$, given by $f(a) = a^2$ for all $a \in \mathbb{R}$.

Example. Let $f(x) = x^2 \in \mathbb{Z}_7[x]$. Then

$$f([0]) = [0]$$
$$f([1]) = f([-1]) = [1]$$
$$f([2]) = f([-2]) = [4]$$
$$f([3]) = f([-3]) = [2].$$

Example (An important example!)**.** Let p be a prime and let $f(x) = x^p \in \mathbb{Z}_p[x]$. We have $f([0]) = [0]$. For $[a] \neq [0]$, Fermat's little theorem tells us $f([a]) = [a]^p = [a]$.

So, $f(x) = x^p$ and $g(x) = x$ define the same function from $\mathbb{Z}_p \to \mathbb{Z}_p$, but they are not equal as polynomials (indeed, $\deg(f(x)) = p$ and $\deg(g(x)) = 1$).

Lecture 24 - 07/03 (quiz)

Lecture 25 - 10/03

Definition. Let $f(x) \in \mathbb{F}[x]$. A **root** of $f(x)$ is an element $c \in \mathbb{F}$ so that $f(c) = 0$.

So, as we saw in the last lecture, the roots of $x^3 - [1]$ in $\mathbb{Z}_{19}[x]$ are $[1], [7]$, and $[11]$.
Roots of polynomials turn out to tell us a lot about how the polynomials factor.

Definition. Let $f, g \in \mathbb{F}[x]$. We say f **divides** g, denoted $f \mid g$, if there exists $q \in \mathbb{F}[x]$ so that $fq = g$.

In $\mathbb{Z}_5[x]$, for example, $x^2 + [4] = (x + [4])(x + [1])$, so $x + [4] \mid x^2 + [4]$.

Now, a useful proposition.

Proposition 52. *Let $f(x) \in \mathbb{F}[x]$ and $c \in \mathbb{F}$. Then $f(c) = 0$ if and only if $(x - c) \mid f(x)$.*

Proof. By the division algorithm, there exists a polynomial $q(x) \in \mathbb{F}[x]$ and a constant (a degree-0 polynomial or the zero polynomial) d so that

$$f(x) = (x - c)q(x) + d.$$

If $f(c) = 0$, then $0 = (c - c)q(x) + d$ so $d = 0$ and $(x - c) \mid f(x)$. Conversely, if $(x - c) \mid f(x)$ then $(x - c)h(x) = f(x)$, and by the uniqueness in the division algorithm, $q(x) = h(x)$ and $d = 0$. Therefore $f(c) = (c - c)q(c) = 0$. ∎

In Assignment 2 you proved that if $x^2 = [a]$ had a solution in $\mathbb{Z}_p$, then there were exactly 2 solutions. However, not every choice of $[a]$ has a solution (for example if $[a] = [2]$ and $p = 5$). So the polynomial $x^2 - [a] \in \mathbb{Z}_p[x]$ has at most 2 roots. In a previous example, we saw that $x^3 - [1] \in \mathbb{Z}_{11}[x]$ has exactly 3 solutions. The fundamental theorem of algebra tells us that a degree d polynomial in $\mathbb{C}[x]$ has exactly d roots (if counted with multiplicity). In $\mathbb{R}[x]$, a degree d polynomial has at most d solutions.

Theorem 53. *Let $f(x) \in \mathbb{F}[x]$ be such that $f(x)$ is not the zero polynomial. Then $f(x)$ has at most $\deg(f(x))$ roots.*

Proof. We will proceed by induction on the degree of f. If $\deg(f) = 0$, then $f = c \neq 0$ for some constant $c \in \mathbb{F}$, so f has no roots.

Assume now that if $\deg(g) = k$, then there are at most k roots of g. Let $f \in \mathbb{F}[x]$ be such that $\deg(f) = k + 1$. If f has no roots, we're done. Suppose f has a root c. Then $f(x) = (x - c)g(x)$ for some $g(x) \in \mathbb{F}[x]$. Since $\deg(f) = 1 + \deg(g)$ we know $\deg(g) = k$. By the inductive hypothesis, g has at most k roots. Now suppose d is a root of f other than c. Then $0 = f(d) = (d - c)(g(d))$. Since $d - c \neq 0$ and since $\mathbb{F}$ is a field, $g(d) = 0$, so d is a root of g. Therefore all roots other than c of f are roots of g and there are at most $k + 1$ roots of f. ∎

Note that in this theorem, we are not counting the roots with multiplicity. However, the theorem would still be true even if we count the roots with multiplicity (see if you can prove this)!

Exercise. Construct a cubic polynomial in $\mathbb{Z}_7[x]$ with exactly 2 roots.

We now have what we need to return to our study of primitive roots.

7.4 Back to primitive roots

The goal now is to show that $\mathbb{Z}_p^*$ has a generator when p is prime. In fact, we can prove something stronger. Not only is it the case that there is an element with order $p - 1$, but for every d dividing $p - 1$, there are $\varphi(d)$ elements of order d in $\mathbb{Z}_p^*$.

Let's take $\mathbb{Z}_{11}^*$ as an example. Here is the power table for $\mathbb{Z}_{11}^*$.

$\mathbb{Z}_{11}^*$	1	2	3	4	5	6	7	8	9	10
x^1	1	2	3	4	5	6	7	8	9	10
x^2	1	4	9	5	3	3	5	9	4	1
x^3	1	8	5	9	4	7	2	6	3	10
x^4	1	5	4	3	9	9	3	4	5	1
x^5	1	10	1	1	1	10	10	10	1	10
x^6	1	9	3	4	5	5	4	3	9	1
x^7	1	7	9	5	3	8	6	2	4	10
x^8	1	3	5	9	4	4	9	5	3	1
x^9	1	6	4	3	9	2	8	7	5	10
x^{10}	1	1	1	1	1	1	1	1	1	1

The possible orders of elements in $\mathbb{Z}_{11}^*$ are $1, 2, 5$, and 10, and of course, we can read off the orders straight from the power table. Since we have the power table in front of us, we can even see that $[2]$ is a generator. Let's write out the elements of $\mathbb{Z}_{11}^*$ as powers of $[2]$ and keep track of their orders.

$$\text{ord}([2]^0) = \text{ord}([1]) = 1$$
$$\text{ord}([2]^1) = \text{ord}([2]) = 10$$
$$\text{ord}([2]^2) = \text{ord}([4]) = 5$$
$$\text{ord}([2]^3) = \text{ord}([8]) = 10$$
$$\text{ord}([2]^4) = \text{ord}([5]) = 5$$
$$\text{ord}([2]^5) = \text{ord}([10]) = 2$$
$$\text{ord}([2]^6) = \text{ord}([9]) = 5$$
$$\text{ord}([2]^7) = \text{ord}([7]) = 10$$
$$\text{ord}([2]^8) = \text{ord}([3]) = 5$$
$$\text{ord}([2]^9) = \text{ord}([6]) = 10$$

We know $[2]$ has order 10, and the important thing to notice here, is that the other powers of $[2]$ that have order 10 are precisely those powers $[2]^k$ such that $\gcd(10, k) = 1$. In particular, there are exactly $\varphi(10)$ such elements. If we look at the elements of order 5, these correspond to powers $[2]^k$ such that $\gcd(k, 10) = 2$, and there are exactly $\varphi(5)$ such elements. This should start to feel like the arguments we used in Lemma 39! Similarly, there are $\varphi(2)$ elements of order 2, and $\varphi(1)$ elements of order 1.

Let's try to prove some of these observations in general.

Lecture 26 - 12/03

Lemma 54. *Let $[a] \in \mathbb{Z}_n^*$ and suppose $\text{ord}([a]) = t$. Then $\text{ord}([a]^k) = t$ if and only if $\gcd(k, t) = 1$.*

Proof. Let $\gcd(k, t) = d$ and suppose $\text{ord}([a]^k) = t$. We have

$$([a]^k)^{\frac{t}{d}} = ([a]^t)^{\frac{k}{d}} = [1].$$

If $d > 1$, then $\frac{t}{d} < t$, contradicting the assumption that $\text{ord}([a]^k) = 1$. Therefore $\gcd(k, t) = 1$.

49

Conversely, suppose $\gcd(k,t) = 1$, so there exist $u, v \in \mathbb{Z}$ so that $ku + tv = 1$. Note that $([a]^k)^t = ([a]^t)^k = [1]$ so $\mathrm{ord}([a]^k) \mid t$. Let $s = \mathrm{ord}([a]^k)$, so in particular $[a]^{ks} = [1]$. Then

$$[a]^s = [a]^{(ku+tv)s} = ([a]^{ks})^u([a]^t)^{vs} = 1.$$

Since $\mathrm{ord}([a]) = t$, we have $t \mid s$ so we conclude $t = \mathrm{ord}([a]^k)$. ■

We can now use this lemma to count exactly now many elements of every possible order there are in $\mathbb{Z}_p^*$. As a consequence we will get that $\mathbb{Z}_p^*$ has a generator for all primes p.

Theorem 55. *Let p be a prime. Then $\mathbb{Z}_p^*$ has $\varphi(d)$ elements of order d for every positive divisor d of $p - 1$.*

Proof. Let $\Omega_d = \{[a] \in \mathbb{Z}_p^* : \mathrm{ord}([a]) = d\}$. Since every element in $\mathbb{Z}_p^*$ has an order dividing $p - 1$ we have

$$\sum_{d \mid p-1} |\Omega_d| = p - 1$$

(recall $\sum_{d \mid p}$ means the sum is taken over all positive divisors d of $p - 1$). By Theorem 40 we have

$$\sum_{d \mid p-1} \varphi(d) = p - 1.$$

Putting these together we get

$$\sum_{d \mid p-1} (\varphi(d) - |\Omega_d|) = 0.$$

If we can show $\varphi(d) \geq |\Omega_d|$ for all d, then we will have a sum of nonnegative integers being equal to 0, and we can conclude that $\varphi(d) = |\Omega_d|$. So let's do that.

If there is no element of order d, then $|\Omega_d| = 0$ and $\varphi(d) \geq |\Omega_d|$. So let's assume $[a] \in \mathbb{Z}_p^*$ has order d. Now, $([a]^k)^d = ([a]^d)^k = [1]$, so every power of $[a]$ is a root of $f(x) = x^d - [1] \in \mathbb{Z}_p[x]$. Since $\mathrm{ord}([a]) = d$, the d distinct elements $\{[a]^0, [a]^1, \ldots, [a]^{d-1}\}$ are roots of $f(x)$. Since $\deg(f(x)) = d$, this is a complete set of roots of $f(x)$ (by Theorem 53).

Every element of order d in $\mathbb{Z}_p^*$ is a root of $x^d - [1]$, so every element of order d is a power of $[a]$. By Lemma 54, there are exactly $\varphi(d)$ powers of $[a]$ that have order d, so $\varphi(d) = |\Omega_d|$.

We may now conclude $\varphi(d) = |\Omega_d|$ for all positive divisors d of $p - 1$. ■

If we take this theorem and focus on the elements of order $p - 1$ we get the following important result.

Corollary 56. *Let p be a prime. There exists a generator of $\mathbb{Z}_p^*$.*

Exercise. Find composite positive integers n and m so that $\mathbb{Z}_n^*$ has a generator and $\mathbb{Z}_m^*$ does not.

Let's exploit this result as best as we can. You may have noticed the curious thing that sometimes -1 has a square root in $\mathbb{Z}_p$. More precisely, for some primes p, there exists an element $[a]$ so that $[a]^2 = [-1]$ in $\mathbb{Z}_p$.

For example, $[2]^2 = [-1]$ in $\mathbb{Z}_5$, but there is no square root of $[-1]$ in $\mathbb{Z}_7$. We are now in a position to completely classify which primes p admit a square root of -1 in $\mathbb{Z}_p$. Another way of phrasing the existence of a square root is to say the polynomial $x^2 + [1]$ has a root in $\mathbb{Z}_p$.

Proposition 57. *Let p be an odd prime. The polynomial $x^2 + [1] \in \mathbb{Z}_p[x]$ has a root if and only if $p \equiv 1 \pmod 4$.*

Proof. Suppose $[a]$ is a root of $x^2 + [1]$. Then $[a]^2 = [-1]$ so $[a]^4 = [1]$. Since $[a] \neq [1]$ and $[a]^2 \neq [1]$, $\text{ord}([a]) = 4$. Therefore $4 \mid p - 1$.

Conversely, suppose $p \equiv 1 \pmod 4$, so $4 \mid p - 1$. By Theorem 55, there are $\varphi(4) = 2$ elements of order 4, call them $[b]$ and $[c]$. Then $[b]^4 = [1]$ so $([b]^2)^2 = [1]$. Since $\text{ord}([b]) = 4$, $[b]^2 \neq [1]$ so $\text{ord}([b]^2) = 2$. However, Theorem 55 tells us there is exactly one element of order 2, and that element is $[-1]$. Therefore $[b]^2 = [-1]$. ∎

Neat! Let's keep pushing.

Proposition 58. *There are infinitely many primes p such that $p \equiv 1 \pmod 4$.*

Proof. Suppose towards a contradiction that $\{p_1, p_2, \ldots, p_k\}$ is the set of primes congruent to 1 (mod 4). Consider $m = (2p_1p_2 \cdots p_k)^2 + 1$, and let q be a prime divisor of m. Note $q \neq p_i$ for all i, and q is an odd prime. Then $(2p_1p_2 \cdots p_k)^2 \equiv -1 \pmod q$, so $q \equiv 1 \pmod 4$, a contradiction. ∎

The main result in this section is that $\mathbb{Z}_p^*$ has a generator when p is prime. It turns out that this is not the only time this happens. Indeed, if you go back to your power tables, $\mathbb{Z}_4^*$ and $\mathbb{Z}_9^*$ also have generators, and so does $\mathbb{Z}_{10}^*$. The complete story, which we shall not prove in this course, is that $\mathbb{Z}_n^*$ has a generator if and only if $n = 1, 2, 4, p^k$, or $2p^k$ where p is an odd prime and $k \in \mathbb{Z}_{>0}$.

Lecture 27 - 14/03

8 The Gaussian Integers

One of the most amazing things in number theory, and indeed in mathematics, is that to solve problems in a particular area, you often need to expand your scope of investigation.

The problem we will be focused on is the following. A **lattice square** is a square in $\mathbb{R}^2$ with the property that the coordinates of its vertices are integers. What are the possible areas of lattice squares? An innocent enough question.

So, for example, there is a lattice square with vertices $(0, 0), (0, 1), (1, 0), (1, 1)$, and another with vertices $(2, 1), (1, -2), (-1, 2), (-2, -1)$. The former square has area 1, and the latter has area 5. In fact, the side length of a lattice square will always be of the form $\sqrt{a^2 + b^2}$ for some $a, b \in \mathbb{Z}$, and so the area of such a square will be an integer of the form $a^2 + b^2$.

We can therefore rephrase our question as which integers n can be written as

$$n = a^2 + b^2$$

where a and b are integers. More succinctly: Which integers are the sum of two squares?

Exercise. Prove that an integer n is a sum of two squares if and only if n is the area of a lattice square.

There are a few ways to answer this question, but the most scenic and intriguing must surely be through the Gaussian integers - an analogue of the integers inside the complex numbers.

Definition. The **Gaussian integers** are the set $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} : a, b \in \mathbb{Z}\}$.

Since the Gaussian integers live in $\mathbb{C}$, they inherit addition and multiplication from $\mathbb{C}$. Explicitly, for $z = a + bi$ and $w = c + di$ in $\mathbb{Z}[i]$ we have

$$z + w = (a + c) + (b + d)i \quad \text{and} \quad zw = (ac - bd) + (ad + bc)i.$$

One way to remember how these operations work is to just do regular addition and multiplication, but wherever you see i^2, replace it with -1. The set $\mathbb{Z}[i]$ endowed with the addition and multiplication from $\mathbb{C}$ forms a ring, but not a field.

Exercise. Show that i is a unit in $\mathbb{Z}[i]$, but $1 + i$ is not.

Let $z = a + bi \in \mathbb{C}$. Recall that the real part of z is $\mathrm{Re}(z) = a$, and the imaginary part of z is $\mathrm{Im}(z) = b$.

Just like the integers $\mathbb{Z}$ lie on the number line, it is often helpful to have a geometric picture of $\mathbb{Z}[i]$. A useful geometric picture is to have $\mathbb{Z}[i]$ lying on the **Gaussian plane**. The Gaussian plane is a way of plotting complex numbers on $\mathbb{R}^2$. The complex number $z = a + bi \in \mathbb{Z}$ is plotted at the point $(a, b) \in \mathbb{R}^2$.

Exercise. Remind yourself how addition and multiplication work geometrically on the Gaussian plane. If you haven't seen it before, plot out a bunch of examples, make a conjecture, and prove it!

Just like the integers have the absolute value, and polynomials have degree, the Gaussian integers have a notion of size.

Definition. Let $z = a + bi$. The **complex conjugate** of z is $\bar{z} = a - bi$. The **norm** of z is $N(z) = z\bar{z} = a^2 + b^2$.

If we denote $|z|$ to be the distance from the origin to z in the Gaussian plane, then $N(z) = |z|^2$.

Recall from solving Pell's equation, that we made use of a different norm. The important thing about that norm was that it was multiplicative. The same thing is true here.

Proposition 59. *Let $z, w \in \mathbb{Z}[i]$. Then*

- $N(z) = 0$ *if and only if* $z = 0$.

- $N(zw) = N(z)N(w)$.

Proof. This is left as an exercise. ∎

Staring at the norm for a while may give you a glimpse as to why the Gaussian integers are important when answering the question of which integers are sums of two squares. Indeed, suppose $n = a^2 + b^2$ and $m = c^2 + d^2$. Then $n = N(a + bi)$ and $m = N(c + di)$. Since the norm plays nicely with multiplication we have

$$nm = N((a + bi)(c + di)) = (ac - bd)^2 + (ad + bc)^2.$$

What we have proved is that if n and m are sums of two squares, then so is nm. This argument should be remeniscent of how we created new solutions out of old ones for Pell's equation!

8.1   Number theory in the Gaussian Integers

Since $\mathbb{Z}[i]$ has addition and multiplication, and a notion of size, the goal is to develop analogues of our theorems for $\mathbb{Z}$. The ultimate aim is to develop a theory of primes and factorisation into primes in $\mathbb{Z}[i]$. To that end, let's begin at the beginning.

Definition. Let $z, w \in \mathbb{Z}[i]$. We say z **divides** w and write $z \mid w$ if there exists $q \in \mathbb{Z}[i]$ such that $zq = w$.

In the integers, the elements 1 and -1 were special when it came to divisibility, because $1 \mid a$ and $-1 \mid a$ for all $a \in \mathbb{Z}$. This is a reflection of the fact that the only units in $\mathbb{Z}$ are ± 1. That is, $\mathbb{Z}^* = \{-1, 1\}$.

Proposition 60. *The group of units in the Gaussian integers is $\mathbb{Z}[i]^* = \{1, -1, i, -i\}$.*

Proof. First note that $1^2 = 1$, $(-1)^2 = 1$, and $i(-i) = 1$, so the elements $1, -1, i, -i$ are indeed units. Suppose now that $z \in \mathbb{Z}[i]^*$, so $zw = 1$ for some $w \in \mathbb{Z}[i]$. Then $N(z)N(w) = N(1) = 1$. Since $N(z), N(w) \in \mathbb{Z}_{\geq 0}$, we must have $N(z) = N(w) = 1$. The only $z \in \mathbb{Z}[i]$ with $N(z) = 1$ are $1, -1, i, -i$, completing the proof. $\blacksquare$

Exercise. Prove that $u \in \mathbb{Z}[i]$ has the property that $u \mid z$ for all $z \in \mathbb{Z}[i]$ if and only if $u \in \mathbb{Z}[i]^*$.

Recall that in the integers, if $a \nmid b$, we can still investigate the relationship between a and b through the division algorithm. The division algorithm in the integers relies on the fact that there is a notion of size on the integers, namely the usual ordering. In $\mathbb{Z}[i]$, we also have a notion of size, and just like with $\mathbb{F}[x]$, this opens the door to a division algorithm.

Lecture 28 - 17/03

Theorem 61 (The division algorithm for the Gaussian integers). . *Let $z, w \in \mathbb{Z}[i]$ with $w \neq 0$. Then there exists $q, r \in \mathbb{Z}[i]$ so that $z = qw + r$ and $N(r) < N(w)$.*

Notice that there is no mention of uniqueness here, and indeed, the q and r are not unique!

The idea of the proof is very similar in spirit to that for $\mathbb{Z}$. As an example, suppose $w = 2 + i$. The set of multiples of w in $\mathbb{Z}[i]$ is $\{\alpha \in \mathbb{Z}[i] : \alpha = \beta w \text{ for some } \beta \in \mathbb{Z}[i]\}$. Let's plot these multiples on the Gaussian plane.
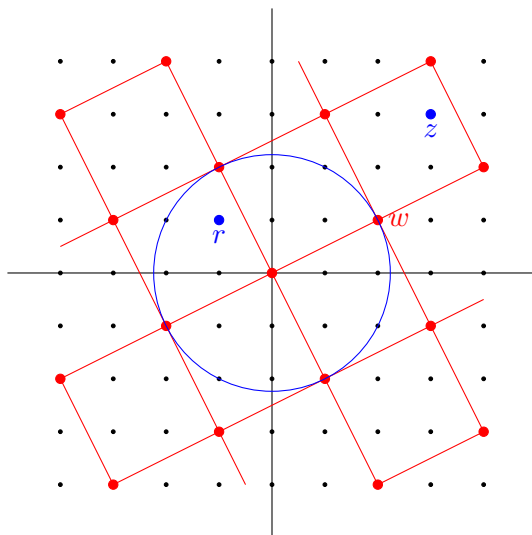
The multiples of $2 + i$ tesselate the Gaussian plane by squares shown above. Now, given any Gaussian integer z, it lies in one of these planes. By taking z and adding multiples of $2 + i$, we can move the point to lie within one of the four squares adjacent to 0, and lie within the blue circle. The Gaussian integers inside the blue circle are precisely those with norm less than $N(2 + i) = 5$. For example, if $z = 3 + 3i$ (indicated by the thick blue dot), we have that $z + (-2)(2 + i) = -1 + i$ lies inside the blue circle (also indicated by a thick blue dot). Therefore $z = 2w + r$ where $r = -1 + i$ and $N(r) = 2 < 5$.

This example also exhibits the non-uniqueness of the division algorithm. By moving z around by different multiples of $2 + i$ we can get three different remainders r satisfying the condition $N(r) < 5$. One of them is $r = -1 + i$. Can you find the other two?

That's the geometric picture, but sometimes we may not want to draw out the Gaussian plane, especially when the Gaussian integers in question are large. In practice, we can perform the division algorithm algebraically much like we do in the integers.

With the integers, suppose we want to divide 13 by 4 and find the remainder. So we want to find $q, r \in \mathbb{Z}$ so that $13 = 4q + r$ and $0 \le r < 4$. What we do is find the floor of $\frac{13}{4}$ (in this case, 3) and set that to be q. This gives $13 = 3 \cdot 4 + r$. Then it's easy to figure out what r is! We'll do the same thing in $\mathbb{Z}[i]$.

Suppose $w = 2 + i$ and $z = 3 + 3i$. Then

$$\frac{z}{w} = \frac{3 + 3i}{2 + i} = \frac{(3 + 3i)(2 - i)}{5} = \frac{9}{5} + \frac{3}{5}i.$$

Now we choose the closest integers to the coefficients, and make those the coefficients of our Gaussian integer q. In this case, $q = 2 + i$. This gives

$$r = 3 + 3i - (2 + i)(2 + i) = -i.$$

So we have $z = qw + r$ where $q = 2 + i$ and $r = -1$. Sure enough, $N(r) < N(w)$.

Exercise. Find Gaussian integers q and r so that $3 + 4i = q(1 + i) + r$ and $N(r) < 2$.

Exercise. What does the division algorithm for $\mathbb{Z}[i]$ say if we restrict our attention to Gaussian integers with imaginary part 0?

8.2 Greatest common divisors

We have a division algorithm, so it's natural to want a Euclidean algorithm in $\mathbb{Z}[i]$. We can always keep applying the division algorithm until we can do so no longer, but what does it give us? Hopefully some kind of greatest common divisor. It will, but there is a twist.

Example. Let $z = 2 + 2i$ and $w = -4 + 2i$. We certainly have that $2 \mid z$ and $2 \mid w$. Is this the Gaussian integer with the largest norm that divides both w and z? Let's check.

We have $N(z) = 8$, $N(w) = 20$, and $N(2) = 4$. Since $\gcd(20, 8) = 4$, we know that the norm of any common divisor of w and z must be at most 4. Therefore 2 has the maximum possible norm of any Gaussian integer that is a common divisor of z and w (to justify this more rigorously, see the next exercise). However, notice that

$$2i(1 - i) = z \quad \text{and} \quad 2i(1 + 2i) = w.$$

So $2i$ is also a common divisor, and $N(2i) = N(2) = 2$.

Exercise. Suppose α is a common divisor of z and w in $\mathbb{Z}[i]$. Prove that $N(\alpha) \mid \gcd(N(z), N(w))$.

So curiously, there doesn't seem to be a unique greatest common divisor. This kind of thing actually happens in the integers also! Note that 2 is the greatest common divisor of 4 and 6, but -2 is also a common divisor. If we consider the absolute value to be the measure of size, then both 2 and -2 have the same size! In the integers we get around this by using the usual ordering on the integers. However, in $\mathbb{Z}[i]$ there is no "usual ordering". To get around this, we must allow for multiplication by units.

Definition. Let $z, w \in \mathbb{Z}[i]$. We say z and w are **associates** if there exists a unit $u \in \mathbb{Z}[i]^*$ such that $z = uw$.

In the example above, the two "greatest common divisors" of $2 + 2i$ and $-4 + 2i$ are associates. This is not a coincidence, as we'll see below.

Definition. Let $w, z \in \mathbb{Z}[i]$, with $w \neq 0$ or $z \neq 0$. A **greatest common divisor** of z and w is a common divisor $d \in \mathbb{Z}[i]$ of maximum norm (that is if c is another common divisor, then $N(d) \geq N(c)$). We write $\gcd(z, w)$ to denote a greatest common divisor.

From the example above, we see $\gcd(z, w) = 2$ or $2i$. The notation is a little difficult because greatest common divisors are not unique, but when we write $\gcd(z, w)$, we take that to represent any greatest common divisor.

Although it is not unique, we will see a little further down that if c and d are greatest common divisors of z and w, then c and d are associates.

Exercise. Compute a greatest common divisor $3 + 4i$ and $1 + i$.

It turns out that the Euclidean algorithm works in $\mathbb{Z}[i]$ (that is it terminates with a greatest common divisor). Not only can you perform Euclid, but you can perform Dilcue! The most valuable part of all of this is that Bezout's identity holds in $\mathbb{Z}[i]$. The proof is also very similar!

Theorem 62 (Bezout's identity in the Gaussian integers)**.** *Let $z, w \in \mathbb{Z}[i]$, not both zero, and let d be a greatest common divisor of z and w. Then there exist $u, v \in \mathbb{Z}[i]$ satisfying $d = zu + wv$.*

Proof. Let $S = \{zu + wv \in \mathbb{Z}[i] : u, v \in \mathbb{Z}[i]\}$, and let δ be a non-zero element in S with the smallest possible norm. Such an element exists since $N(a) \in \mathbb{Z}_{\geq 0}$ for all $a \in \mathbb{Z}[i]$.

Let $\delta = zu_0 + wv_0$. Our goal is to show δ is an associate of d. To do that, we will first show that δ is a greatest common divisor of z and w.

To see δ is a common divisor of z and w, write $z = q\delta + r$ with $N(r) < N(\delta)$ as per the division algorithm. Then

$$r = z - q\delta = z - q(zu_0 + wv_0) = z(1 - qu_0) + w(qv_0).$$

Therefore $r \in S$ and since δ is an element of minimal norm in S and $N(r) < N(\delta)$, we must have $N(r) = 0$ so $r = 0$. Alas, $z = q\delta$ and $\delta \mid z$. A similar argument gives $\delta \mid w$.

Suppose now that c is another common divisor, and write $z = c\alpha$ and $w = c\beta$ for $\alpha, \beta \in \mathbb{Z}[i]$. Then $\delta = zu_0 + wv_0 = c(\alpha u_0 + \beta v_0)$ so $c \mid \delta$. Therefore, $N(c) \mid N(\delta)$. Since the norm takes values in non-negative integers, it follows that $N(c) \leq N(\delta)$, so δ is a greatest common divisor.

Now, since $d \mid z$ and $d \mid w$, we have $d \mid zu_0 + wv_0$ and so $d \mid \delta$. Write $d\gamma = \delta$ for $\gamma \in \mathbb{Z}[i]$. Then $N(d)N(\gamma) = N(\delta)$ and since $N(d) = N(\delta) \neq 0$, we must have $N(\gamma) = 1$. Therefore $\gamma \in \mathbb{Z}[i]^*$ (so d and δ are associates). We now have

$$d = \gamma^{-1}\delta = z(\gamma^{-1}u_0) + w(\gamma^{-1}v_0)$$

completing the proof. ∎

Just like with the integers, to find gaussian integers u and v satisfying $zu + wv = \gcd(z, w)$, we perform the Euclidian algorithm forwards and then backwards!

Exercise. Compute a greatest common divisor of $3 + i$ and $-3 - 4i$.

Using the latter part of the proof of Bezout's identity, we have the following corollary.

Corollary 63. *Let $z, w \in \mathbb{Z}[i]$, not both zero. If c and d are greatest common divisors of z and w, then c and d are associates.*

Proof. This is an exercise. See if you can extract the argument from the proof of Bezout's identity above. ∎

Our goal is to talk about primes in $\mathbb{Z}[i]$, and in particular to prove the following important property (which you should recall is true in the integers): If π is prime in $\mathbb{Z}[i]$ and $\pi \mid zw$, then $\pi \mid z$ or $\pi \mid w$. Once we have this, we will also have a theorem about unique factorisation into primes in $\mathbb{Z}[i]$.

Before we do that, or indeed before we define what a prime is in $\mathbb{Z}[i]$, we must first talk about coprimeness.

Definition. We say that $z, w \in \mathbb{Z}[i]$ are **coprime** if $\gcd(z, w) = 1$.

Recall that this definition really says that $\{1, -1, i, -i\}$ are all the greatest common divisors of z and w.

Example. The Gaussian integers $z = 1 - 2i$ and $w = 1 + 2i$ are coprime. To see this, let d be a common divisor of both z and w. Then $d \mid z + w$ so $d \mid 2$. Since $N(2) = 4$ we must have $N(d) \mid 4$. However, we also have that $N(d) \mid N(z)$ so $N(d) \mid 5$. Since $N(d)$ is a common divisor of 4 and 5, it must be the case that $N(d) = 1$. Therefore d is a unit in $\mathbb{Z}[i]$ and we conclude that z and w are coprime.

Just like in the integers, once we have Bezout's identity, we get the following super useful proposition.

Proposition 64. *Let $z, w \in \mathbb{Z}[i]$. Then z and w are coprime if and only if there exist $u, v \in \mathbb{Z}[i]$ such that $zu + wv = 1$.*

Proof. This is left as an exercise. ∎

The previous Proposition is a wonderfully powerful characterisation of being coprime, as we have seen time and time again in this course. In fact, here is one of the more useful properties of coprime integers, transferred over to coprime Gaussian integers.

Proposition 65. *Let $z, w, \gamma \in \mathbb{Z}[i]$ be such that z and w are coprime. If $z \mid w\gamma$, then $z \mid \gamma$.*

Proof. We imitate the proof from the integers. Since $z \mid w\gamma$, write $z\delta = w\gamma$. Since z and w are coprime there exist $u, v \in \mathbb{Z}[i]$ satisfying $zu + wv = 1$. Multiplying by γ gives $z\gamma u + w\gamma v = \gamma$. Therefore $z(\gamma u + \delta v) = \gamma$, and $z \mid \gamma$. ∎

8.3 Gaussian primes

We are finally ready to talk about primes in $\mathbb{Z}[i]$, which we call Gaussian primes. While it is tempting to say that a prime is something whose only divisors are 1 and itself, there are no Gaussian integers with this property! This is because of all those pesky units, which divide every Gaussian integer! Given any Gaussian integer z, we automatically have eight divisors: $1, -1, i, -i, z, -z, iz, -iz$. Notice that the story is in parallel with the integer story that every $n \in \mathbb{Z}$ has four divisors: ± 1 and $\pm n$. We can state this fact a little more succinctly by saying all the associates of 1 and all the associates of z are divisors of z. These are the divisors we get for free! With this in mind, we make the following definition of a prime element in $\mathbb{Z}[i]$.

Definition. Let π be a Gaussian integer that is not a unit. We say π is a **Gaussian prime** if $z \mid \pi$ implies that $z \in \mathbb{Z}[i]^*$ or z is an associate of π.

Let's find some Gaussian primes!

Example. Consider $\pi = 1 + i$. Then $N(\pi) = 2$. Now suppose $z \mid \pi$. Then $N(z) \mid N(\pi)$, so $N(z) = 1$ or $N(z) = 2$. If $N(z) = 1$, then z is a unit. If $N(z) = 2$, there are only four possibilities. They are $\{1 + i, -1 + i, 1 - i, -1 - i\}$ (try to justify this for yourself!). All of these are associates of π. Therefore π is a Gaussian prime.

Example. Let $\pi = 1 + 2i$, so $N(\pi) = 5$. Suppose $z \mid \pi$, so $N(z) = 1$ or 5. Again, if $N(z) = 1$, $z \in \mathbb{Z}[i]^*$. To show π is prime, it remains to show that if $N(z) = 5$, then z is an associate of π. It is tempting to do what we did in the previous example, and just check all Gaussian integers with norm 5. However, there are Gaussian integers with norm 5 that *are not* associates of $1 + 2i$. For example, $1 - 2i$. However, it turns out that $1 - 2i$ is not a divisor of $1 + 2i$, so all is not lost! By writing down all Gaussian integers of norm 5, you can check that the only ones that divide π are precisely the associates of π.

Since I don't want to write down all eight (so many!!!) Gaussian integers with norm 5, let's just prove this directly. Suppose $zw = \pi$ and $N(z) = 5$. Then $5N(w) = 5$. Therefore, $N(w) = 1$ and $w \in \mathbb{Z}[i]^*$. Alas, z is an associate of π and π is a Gaussian prime.

The previous example relied on the following useful exercise.

Exercise. Suppose $z \mid w$ and $N(z) = N(w)$. Then z and w are associates.

With this exercise in our back pockets, and armed with the fact that $N(zw) = N(z)N(w)$, we have a quick way of identifying some Gaussian primes.

Proposition 66. *Let $z \in \mathbb{Z}[i]$ be such that $N(z)$ is a prime in the integers. Then z is a Gaussian prime.*

Proof. Suppose $N(z) = p$ where p is a prime in $\mathbb{Z}$. Let $w \in \mathbb{Z}[i]$ be such that $w \mid z$. Then $N(w) = 1$ or p. If $N(w) = 1$, then $w \in \mathbb{Z}[i]^*$. If $N(w) = p = N(z)$, then w and z are associates. Alas, z is a Gaussian prime. ∎

Great, so we automatically know that $1 + i$, $1 + 2i$, $3 + 2i$, $4 + i$, and $12 + 13i$ (and all their associates) are Gaussian primes! Indeed, note that $12^2 + 13^2 = 313$ which is an integer prime.

It will turn out to be essential to specify whether or not an integer a is a Gaussian prime or an integer prime. Sometimes integer primes are referred to as **rational primes**, and we may use either term.

Let's now prove a very important property about Gaussian primes, which we will use multiple times for the rest of this section. This property is shared by integer primes (see Proposition 11) and is commonly known as Euclid's lemma.

Theorem 67 (Euclid's lemma for Gaussian primes). *Let π be a Gaussian prime and suppose $\pi \mid zw$ for some $z, w \in \mathbb{Z}[i]$. Then $\pi \mid z$ or $\pi \mid w$. More generally, if $\pi \mid z_1 z_2 \cdots z_k$, then $\pi \mid z_j$ for some j.*

Proof. Suppose $\pi \nmid z$. Let d be a common divisor of π and z. Since π is a Gaussian prime, d is either a unit or an associate of π. Since $\pi \nmid z$, no associate of π is a divisor of z. Therefore d is a unit so π and z are coprime. Therefore, $\pi \mid w$ by Proposition 65. The general statement follows by induction on k and is left as an exercise. ∎

Wonderful! We can now address the question of which (integer) primes can be written as a sum of two squares. Let's try to write some down. For starters we have $2 = 1^2 + 1^2$. We can just check by looking at sums of 0^2, 1^2, and 2^2 that 3 is not the sum of two squares. For the next prime, we have $5 = 2^2 + 1^2$.

The fact that 3 is not the sum of two squares is a reflection of a more general phenomenon, that reveals itself when looking at things mod 4. For any $a \in \mathbb{Z}$, $a^2 \equiv 0$ or 1 (mod 4). Therefore, we know that a sum of two squares is congruent to either 0, 1, or 2 (mod 4). So we immediately know that 3, 7, 11, and every prime that is congruent to 3 (mod 4) cannot be written as a sum of two squares.

For the integer primes p that are congruent to 1 (mod 4), being able to find $a, b \in \mathbb{Z}$ such that $p = a^2 + b^2$ is equivalent to finding $a + bi \in \mathbb{Z}[i]$ such that $N(a + bi) = p$. The remarkable thing here is that finding such a Gaussian integer is always possible!

The next theorem is known as Fermat's Christmas Theorem, since Fermat wrote about the problem in a letter to Marin Mersenne (of Mersenne prime fame) dated Christmas day in 1640. Although the theorem is affectionately named after Fermat, what we are about to prove was first published by Albert Girard in 1625.

Theorem 68 (Fermat's Christmas Theorem or Girard's Theorem). *Let p be an odd prime. Then p is the sum of two squares if and only if $p \equiv 1$ (mod 4).*

Proof. Since p is an odd prime, it is either congruent to 1 or 3 (mod 4). If $p \equiv 3$ (mod 4), then it cannot be the sum of two squares as discussed above. For the converse, suppose $p \equiv 1$ (mod 4). Then $x^2 + [1] \in \mathbb{Z}_p[x]$ has a root, so there exists $m \in \mathbb{Z}$ satisfying $p \mid m^2 + 1$. In $\mathbb{Z}[i]$ we have $p \mid (m + i)(m - i)$. However, the imaginary part of $\frac{m \pm i}{p}$ is $\frac{1}{p}$, which is not an integer. Therefore $p \nmid (m + i)$ and $p \nmid (m - i)$. We can therefore conclude that p is not a Gaussian prime. Let $z \in \mathbb{Z}[i]$ be a divisor of p that is not a unit or an associate of p. It must be the case that $1 < N(z) < N(p) = p^2$. However, since $z \mid p$, we have $N(z) \mid N(p)$. The only possibility for $N(z)$ is that $N(z) = p$. Writing $z = a + bi$ we have $a^2 + b^2 = p$, completing the proof. ∎

It is certainly worth taking a step back and paying attention to all the ingredients that went into that proof. We needed to know that if $p \equiv 1$ (mod 4), $x^2 + [1] \in \mathbb{Z}_p[x]$ has a root. The existence of such a root relied on generators existing for $\mathbb{Z}_p^*$ whenever p is prime. We needed to venture into the complex plane and study Gaussian integers. We needed know that Euclid's lemma holds for Gaussian primes. We needed properties of the norm on $\mathbb{C}$, which we used many times without mention. Phew!

Remarkably, an odd prime p being the sum of two squares is equivalent to $[-1]$ being a square in $\mathbb{Z}_p$, which is equivalent to $p \equiv 1$ (mod 4). Since we know there are infinitely many primes $p \equiv 1$ (mod 4), we know there are infinitely many primes p that are the sum of two squares!

Exercise. Write 61, 89, and 109 as the sum of two squares. In how many different ways are each of those integer primes expressable as a sum of two squares?

Exercise. Suppose that p is an integer prime and suppose there exist perfect (integer) squares a, b, c, d so that $p = a + b = c + d$. Prove that $\{a, b\} = \{c, d\}$. That is, prove that if a prime is expressable as the sum of two squares, it is uniquely expressable as the sum of two squares. *Hint: Euclid's lemma may come in handy here.*

8.4 The classification of Gaussian primes

Now that we've played around with Gaussian primes a little, we have a natural question. Which elements $a + bi \in \mathbb{Z}[i]$ are Gaussian primes?

We already have some partial results. For example, we know that if $N(z) = p$ where p is an integer prime, then z is a Gaussian prime. The proof of Fermat's Christmas theorem secretly showed that if p is an integer prime such that $p \equiv 1 \pmod 4$, then p is not a Gaussian prime (can you extract an argument for that from the proof?).

For example, 5 is not a Gaussian prime since $5 = (1 + 2i)(1 - 2i)$.

So, curiously, not every integer prime is a Gaussian prime, but some are (like 3). By extending the integers from $\mathbb{Z}$ to $\mathbb{Z}[i]$, we have introduced new ways to factor some previously unfactorable numbers!

Let's begin our classification by investigating the relationship between integer primes and Gaussian primes.

Exercise. Prove that if $p + 0i$ is a Gaussian prime, then p is an integer prime.

Proposition 69. *Let $a \in \mathbb{Z}_{\geq 0}$. Then $a \in \mathbb{Z}[i]$ is a Gaussian prime if and only if a is an integer prime congruent to $3 \pmod 4$.*

Proof. Assume that $a \equiv 3 \pmod 4$ and a is a prime. Let $z \mid a$ in $\mathbb{Z}[i]$. Then $N(z) \mid N(a) = a^2$. Since the only positive integer divisors of a^2 are $1, a$, and a^2, to show that a is a Gaussian prime, it suffices to show that $N(z) \neq a$. Since $a \equiv 3 \pmod 4$, it is not the sum of two integer squares, and therefore it cannot be the norm of a Gaussian integer. Alas, $N(z) \neq a$ and a is a Gaussian prime.

Conversely, suppose a is not an integer prime congruent to $3 \pmod 4$. By the previous exercise, if a is not a prime, then a is not a Gaussian prime. If $a = 2$, then $a = (1 + i)(1 - i)$ so a is not a Gaussian prime. If $a \equiv 1 \pmod 4$, then by Fermat's Christmas Theorem, $a = b^2 + c^2$ for some $b, c \in \mathbb{Z}$. Therefore, $a = (b - ci)(b + ci)$. Since a is a prime, it is not an integer square, and so $b \neq 0$ and $c \neq 0$. Therefore $b + ci \notin \mathbb{Z}[i]^*$ and $b + ci$ is not an associate of a. We can finally conclude a is not a Gaussian prime. ∎

Exercise. Prove that there are infinitely many Gaussian primes.

Okay, let's gather up all the primes we've found so far. We know that if $N(\pi) = p$ for some integer prime p, then π is a Gaussian prime (Proposition 66). We know that if $p \equiv 3 \pmod 4$ is an integer prime, then p is a Gaussian prime (Proposition 69). It turns out that these are all of them!

Theorem 70. *Let π be a Gaussian prime. Then π is either*

- *an associate of $1 + i$,*

- *an associate of an integer prime p where $p \equiv 3 \pmod 4$, or*

- *$N(\pi) = p$ where p is an integer prime such that $p \equiv 1 \pmod 4$.*

Proof. Let π be a Gaussian prime. The first step is to prove that π divides an integer prime. To see this, note that $N(\pi) = \pi\overline{\pi}$. Since $N(\pi)$ is a positive integer, it is a product of integer primes. Therefore, π divides a product of integer primes and so must divide one of them by Euclid's lemma. Let p be such an integer prime.

If $p = 2$, we have $N(\pi) = 4$. If $N(\pi) = 4$, then π would be an associate of 2. However, $2 = (1 + i)(1 - i)$ and so is not prime. So we must have $N(\pi) = 2$. The only Gaussian integers with norm 2 are associates of $1 + i$.

If $p \equiv 3 \pmod 4$, then since p is a Gaussian prime and $\pi \mid p$, we must have that π is an associate of p.

Finally, suppose $p \equiv 1 \pmod 4$. Then $p = a^2 + b^2 = (a + bi)(a - bi)$. Since $\pi \mid p$, $\pi \mid a + bi$ or $\pi \mid a - bi$ by Euclid's lemma. Without loss of generality, write $\pi z = a + bi$. Then $N(\pi)N(z) = N(a + bi) = p$. Since p is an integer prime and $N(\pi) > 1$, it must be that $N(z) = 1$. Therefore $N(\pi) = p$, completing the proof. ∎

So the moral of the story is that if we want to find Gaussian primes, we just need to look at factors of integer primes! The integer primes fall into three cases:

- The case $p = 2$. This gives rise to a single Gaussian prime $1 + i$, and its associates.

- The case $p \equiv 1 \pmod 4$. Each such integer prime can be written as $p = a^2 + b^2$ and gives rise to exactly two Gaussian primes, $a + bi$ and $a - bi$, and their associates. Note that we have not proved that there are exactly two, but it's true (see the exercise below).

- The case $p \equiv 3 \pmod 4$. Each such integer prime (and each of its associates) is a Gaussian prime.

Exercise. Suppose $p \equiv 1 \pmod 4$ is an integer prime. Prove that there exists a Gaussian prime divisor α of p with the property that every prime divisor π of p is an associate of either α or $\overline{\alpha}$.

There are many other properties of integer primes that also hold for Gaussian primes. For example, it is true that every Gaussian integer can be written as a product of Gaussian primes, and such a prime factorisation is unique (after possibly reordering the product and replacing each Gaussian prime by an associate). See if you can state and prove an analogue in $\mathbb{Z}[i]$ of the fundamental theorem of arithmetic.

Now that we know what the primes are, let's do the one thing we are compelled to do: factor!

Example. We have
$$91 = 7 \cdot 13 = 7(3 + 2i)(3 - 2i)$$
which is the Gaussian prime factorisation of $91 \in \mathbb{Z}[i]$.

Exercise. Factor $17 + 4i$ into Gaussian primes.

Lecture 32 - 26/03

A lot of the past chunk of the course has involved, in some way or another, odd integer primes. Let's pause for a moment and put together a bunch of results from the last few sections.

Theorem 71. *Let p be an odd (integer) prime. Then the following are equivalent.*

- $p \equiv 1 \pmod 4$.

- $p = a^2 + b^2$ *for some* $a, b \in \mathbb{Z}$.

- $[-1]$ *is a square in* $\mathbb{Z}_p^*$.

- p *is not a Gaussian prime.*

It's a great exercise to gather up the relevant results from the course to prove this theorem.

8.5 Integers that are sums of two squares

Let's shift our attention now to the more general question of which integers are sums of two squares. One way to think about this problem is to ask which integers are norms of Gaussian integers.

Our first bit of progress into this problem comes from remembering that norms play nicely with multiplication.

Example. we know $5 = 2^2 + 1^2 = N(2+i)$ and $13 = 3^2 + 2^2 = N(3+2i)$. Therefore

$$65 = N(2+i)N(3+2i) = N((2+i)(3+2i)) = N(4+7i) = 4^2 + 7^2.$$

Great! So any product of numbers that are sums of two squares are again sums of two squares. On the other hand, we know that if $n \equiv 3 \pmod 4$, it cannot be the sum of two squares.

Consider now the prime factorisation of some positive integer n. We know 2 is a sum of two squares, as is every (integer) prime $p \equiv 1 \pmod 4$. So if the prime factorisation of n contains only 2 and primes congruent to 1 (mod 4), then n will be the sum of two squares. If there are an odd number of primes congruent to 3 (mod 4) in the prime factorisation, then $n \equiv 3 \pmod 4$, so n will not be able to be written as the sum of two squares.

This leaves us in a situation where $n \equiv 1 \pmod 4$, but n does contain some prime divisors congruend to 3 (mod 4). Let's look at some examples.

Example. We know $35 = 5 \cdot 7 \equiv 3 \pmod 4$ so 35 certainly cannot be written as a sum of two squares.

Example. On the other hand, $245 = 5 \cdot 7^2$ can be written as a sum of two squares since 5 and 7^2 are both sums of squares. More concretely,

$$245 = N(1+2i)N(7) = N(7+14i) = 7^2 + 14^2.$$

This is intriguing because $7 \equiv 3 \pmod 4$, and yet 245 can be written as a sum of two squares!

Example. Consider $7 \cdot 11 = 77 \equiv 1 \pmod 4$. Note that both 7 and 11 are congruent to 3 (mod 4). We can just check all pairs of integers $a, b \le \sqrt{77}$ and conclude that 77 is not a sum of two squares. This works, but doesn't generalise well. Let's give a slightly more general argument.

Suppose $77 = a^2 + b^2$ for some $a, b, \in \mathbb{Z}$. Looking at this in $\mathbb{Z}_7$ we have $[a]^2 + [b]^2 = [0]$. If $[a] \ne [0]$, then $[a]^{-1}$ exists and we have $[-1] = ([b][a]^{-1})^2$. So $[-1]$ is a square and it must be the case $7 \equiv 1 \pmod 4$, a contradiction! Therefore we must have $[a] = 0$, and as a consequence, $[b] = [0]$. However, if $7 \mid a$, then $7^2 \mid a^2$, and similarly $7^2 \mid b^2$. Therefore we must have $7^2 \mid a^2 + b^2 = 77$, which is false! Alas, we can conclude that 77 is not a sum of two squares.

We could have argued similarly for 11 and reached a similar contradiction since $11^2 \nmid 77$.

Theorem 72. *Let $n \in \mathbb{Z}_{>0}$ be such that $n = 2^v \prod p_i^{e_i} \prod q_j^{f_j}$ where each $p_i \equiv 1 \pmod 4$ and $q_i \equiv 3$ (mod 4). Then n is the sum of two squares if and only if every f_j is even.*

Proof. Suppose each f_j is even. Then each q_j^f is a square, and in particular is a sum of two squares (the other being 0^2). Since each p_i is the sum of two squares, and 2 is the sum of two squares, n is the sum of two squares.

Conversely, suppose $n = a^2 + b^2$ for some integers $a, b \in \mathbb{Z}$. We can write $n = k^2 m$ where the prime factorisation of m contains no powers greater than 1. We want to show that if p is a prime and $p \mid m$, then $p \equiv 1 \pmod 4$.

To this end, let $d = \gcd(a, b)$. Then

$$\frac{n}{d^2} = \left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2.$$

Since m has no divisors which are perfect squares, it must be the case that $d^2 \mid k^2$. Writing $d^2 t = k^2$ we get

$$tm = \left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2.$$

Now, suppose $p \mid m$ is a prime. Let $x = \frac{a}{d}$ and $y = \frac{b}{d}$, and note that x and y are coprime integers. Since $p \mid m$ we have, in $\mathbb{Z}_p$, $[0] = [x]^2 + [y]^2$.

Note that $[x] = [0]$ if and only if $[y] = [0]$. However, if $[x] = [y] = [0]$, we must have $p \mid x$ and $p \mid y$, contradicting the fact that x and y are coprime. Therefore $[x] \neq [0]$. Since $\mathbb{Z}_p$ is a field, $[x]^{-1}$ exists and after some rearranging we have $[-1] = ([y][x]^{-1})^2$. Therefore $[-1]$ is a square in $\mathbb{Z}_p$, and $p \equiv 1 \pmod 4$. ∎

Hooray! So we know $5^2 \cdot 7^3 \cdot 11^{13}$ cannot be written as a sum of two squares, but $5^2 \cdot 7^4 \cdot 11^{14}$ can!

8.6 Pythagorean triples

If there is any theorem in mathematics that just about everyone knows, or at least has heard of, it's the Pythagorean theorem. It turns out that the statement is usually recalled as an implication, but it's really and if-and-only-if statement!

Theorem 73 (Pythagoras, probably). *Let a, b, c be positive real numbers that are the side lengths of a triangle. Then $a^2 + b^2 = c^2$ if and only if the sides with lengths a and b form a right angle.*

Cool. This is a very very useful theorem. If you're building something and you want to make sure two pieces of material are at a right angle, you can use this theorem and then shout out to all the people around you "see, math is useful!".

When a child is born into this world, the first right-angled triangle to which they are introduced is the triangle with side lengths 3, 4, and 5. Then they might see $6, 8, 10$, but they quickly realise it's just the 3, 4, 5 triangle in disguise. The next one they see is the 5, 12, 13 triangle, which inevitably blows their little infant minds! Surely there must be more triples of integers like this.

Definition. A **Pythagorean triple** is a triple (a, b, c) of positive integers satisfying $a^2 + b^2 = c^2$. A Pythagorean triple is **primitive** if $\gcd(a, b, c) = 1$.

If we have all primitive Pythagorean triples, then we can get all Pythagorean triples by scaling. So the question now becomes, how do we find all Pythagorean triples?

Let's start with some useful properties.

Proposition 74. *Let (a, b, c) be a primitive Pythagorean triple. Then*

1. $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$.

2. One of a and b is odd, and the other is even.

3. c is odd.

Proof.   1. If d is a common divisor of any two of a, b, and c, then since $a^2 + b^2 = c^2$, d must divide the third. Therefore any common divisor of two of them must be a common divisor of all three. Therefore $d = \pm 1$ and so $\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = 1$.

2. By part 1, we cannot have $a \equiv b \equiv c \equiv 0 \pmod 2$. Since $a^2 + b^2 \equiv c^2 \pmod 2$, at least one of a or b must be odd. If both a and b are odd, then c is even. Also, $a^2 + b^2 \equiv 2 \pmod 4$ and $c^2 \equiv 0 \pmod 4$. Therefore, both a and b cannot be odd. It follows that exactly one of a or b is odd.

3. By part 2, if exactly one of a or b is odd, $c^2 = a^2 + b^2 \equiv 1 \pmod 2$ and thus c is odd. ∎

    We can now parametrise all primitive Pythagorean triples! Once again, our lives are simplified by exploiting the Gaussian integers.

Lemma 75. *Suppose $a^2 + b^2 = c^2$ is a primitive Pythagorean triple. Then $a + bi$ and $a - bi$ are coprime in $\mathbb{Z}[i]$.*

Proof. Suppose there exists a Gaussian prime π that is a common divisor of $a + bi$ and $a - bi$. We will show that it follows that a and b are not coprime integers.

    Note that $(a + bi) + (a - bi) = 2a$ and $-i(a + bi) + i(a - bi) = 2b$ and so $\pi \mid 2a$ and $\pi \mid 2b$. First we will show that π cannot divide 2. If it did, then π is an associate of $1 + i$ and so $N(\pi) = 2$. However, since $\pi \mid a + bi$, $N(\pi) \mid N(a + bi) = c^2$. This is a problem since c^2 is odd, so $N(\pi) \neq 2$. We conclude that $\pi \nmid 2$.

    Since π is a Gaussian prime, $\pi \mid a$ and $\pi \mid b$ by Euclid's lemma. Therefore a and b are not coprime in $\mathbb{Z}[i]$ and thus they are not coprime in $\mathbb{Z}$ (see the exercise following the proof).

    We have shown that if π is a common divisor of $a + bi$ and $a - bi$, then a and b are not coprime in $\mathbb{Z}$. Therefore (a, b, c) cannot be a primitive Pythagorean triple (by Proposition 74(1)), completing the proof. ∎

Exercise. Let $a, b \in \mathbb{Z}$. Prove that a and b are coprime in $\mathbb{Z}$ if and only if a and b are coprime in $\mathbb{Z}[i]$.

    The proof of the main theorem below exploits some ideas from the Gaussian integers, and in particular the next fact, which is an exercise.

Exercise. Let $z, w \in \mathbb{Z}[i]$ be coprime and suppose $zw = \gamma^2$ for some $\gamma \in \mathbb{Z}[i]$. Then $z = u\delta^2$ for some $\delta \in \mathbb{Z}[i]$ and $u \in \mathbb{Z}[i]^*$.

    Great, let's try to find Pythagorean triples! We will look, one last time in this course, in the Gaussian integers.

    Suppose (a, b, c) is a primitive Pythagorean triple, so $c^2 = (a + bi)(a - bi)$. By Lemma 75, $a + bi$ and $a - bi$ are coprime in $\mathbb{Z}[i]$. Then from the previous exercise we can write $a + bi = u(m + ni)^2$ for some unit $u \in \mathbb{Z}[i]^*$. Expanding this out gives

$$a + bi = u(m^2 - n^2 + 2mni)$$

If $u = \pm 1$, then equating real and imaginary parts gives

$$a = \pm(m^2 - n^2) \quad \text{and} \quad b = \pm 2mn.$$

The roles of a and b are switched if $u = \pm i$. So, we have a place to look for Pythagorean triples!

Indeed, let $m, n \in \mathbb{Z}$ and set $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$. Then

$$\begin{aligned}
a^2 + b^2 &= (m^2 - n^2)^2 + 4m^2n^2 \\
&= m^4 + n^4 - 2m^2n^2 + 4m^2n^2 \\
&= (m^2 + n^2)^2 \\
&= c^2.
\end{aligned}$$

Furthermore, we showed above that if you start with a primitive Pythagorean triple, you get integers m and n.

Theorem 76 (Classification of Primitive Pythagorean Triples). *Let (a, b, c) be a primitive Pythagorean triple such that $2 \mid b$. There exist positive coprime integers m and n, one of which is even, such that*

$$a = m^2 - n^2, \quad b = 2mn, \quad \text{and} \quad c = m^2 + n^2.$$

Conversely, if m and n are positive coprime integers, one of which is even, then

$$(m^2 - n^2, 2mn, m^2 + n^2)$$

is a primitive Pythagorean triple.

Proof. We begin with the converse. Before the statement of the theorem, we have verified that $(m^2 - n^2, 2mn, m^2 + n^2)$ is a Pythagorean triple. To see it is primitive, suppose p is a prime divisor of all three integers. Since exactly one of m and n is even, $m^2 + n^2$ is odd, so $p \neq 2$. Since $p \mid 2mn$, we must have $p \mid m$ or $p \mid n$. If $p \mid m$, then since $p \mid m^2 + n^2$, we must have $p \mid n^2$ and so $p \mid n$. We can similarly deduce that if $p \mid n$ then $p \mid m$. However, this contradicts the assumption that $\gcd(m, n) = 1$. Therefore such a prime p cannot exist, and $(m^2 - n^2, 2mn, m^2 + n^2)$ is a primitive Pythagorean triple.

For the forward direction, suppose $a^2 + b^2 = c^2$ with $\gcd(a, b, c) = 1$ and $2 \mid b$. Then $\gcd(a, b) = 1$, and a and c are odd. By Lemma 75, $a + bi$ and $a - bi$ are coprime in $\mathbb{Z}[i]$. Thus, there exists $m + ni \in \mathbb{Z}[i]$ satisfying

$$a + bi = u(m + ni)^2 = u(m^2 - n^2) + 2umni.$$

Suppose m and n have the same parity (they are both even or both odd). Then, depending on the value of $u \in \mathbb{Z}[i]^*$, the real and imaginary parts of $u(m^2 - n^2) + 2umni$ are both even. This cannot happen since a is odd. Therefore, we must have that exactly one of m and n is even. Since b is even, it must be the case that $u = \pm 1$. Therefore, by equating real and imaginary parts we have

$$a = \pm(m^2 - n^2) \quad \text{and} \quad b = \pm 2mn.$$

If necessary, we can now replace m and n by their absolute values to get $2mn > 0$. We can then switch the roles of m and n if necessary to ensure $m^2 - n^2 > 0$. After doing this, we have shown there are positive integers m, n, exactly one of which is even, satisfying

$$a = m^2 - n^2 \quad \text{and} \quad b = 2mn.$$

It remains to show that $\gcd(m, n) = 1$ to complete the proof. To that end, let $d > 1$ be a common prime divisor of m and n. Then $d \mid 2mn = b$ and $d \mid m^2 - n^2 = a$, contradicting our assumption that $\gcd(a, b) = 1$. ∎

Hooray! We can now generate all the primitive Pythagorean triples by running through coprime pairs of positive integers with opposite parities. Here are the first few:

(m, n)	$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$
$(2, 1)$	$(3, 4, 5)$
$(3, 2)$	$(5, 12, 13)$
$(4, 1)$	$(15, 8, 17)$
$(4, 3)$	$(7, 24, 25)$
$(5, 2)$	$(21, 20, 29)$
$(5, 4)$	$(9, 40, 41)$
$(6, 1)$	$(35, 12, 37)$
$(6, 5)$	$(11, 60, 61)$
$(7, 2)$	$(45, 28, 53)$
$(7, 4)$	$(33, 56, 65)$
$(7, 6)$	$(13, 84, 85)$
$(8, 1)$	$(63, 16, 65)$
$(8, 3)$	$(55, 48, 73)$
$(8, 5)$	$(39, 80, 89)$
$(8, 7)$	$(15, 112, 113)$
$(9, 2)$	$(77, 36, 85)$
$(9, 4)$	$(65, 72, 97)$
$(9, 8)$	$(17, 144, 145)$

Cool! If you see anything interesting in the table, write it down and try to prove it. For example:

Exercise. Let a, b, c be a Pythagorean triple. Prove that $3 \mid a$, $3 \mid b$, or $3 \mid c$.

9   That's All Folks

We've barely scratched the surface of the beautiful, deep and mysterious area of mathematics known as number theory. Hopefully you've had a glimpse of the magnificence that lies ahead if you choose to continue further into the woods. Good luck.

A   List of the first 1000 primes

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 97 101 103 107 109 113 127 131
137 139 149 151 157 163 167 173 179 181 191 193 197 199 211 223 227 229 233 239 241 251 257 263
269 271 277 281 283 293 307 311 313 317 331 337 347 349 353 359 367 373 379 383 389 397 401 409
419 421 431 433 439 443 449 457 461 463 467 479 487 491 499 503 509 521 523 541 547 557 563 569
571 577 587 593 599 601 607 613 617 619 631 641 643 647 653 659 661 673 677 683 691 701 709 719
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B Equivalence Relations and Partitions

Equivalence classes and partitions pop up in just about every area of mathematics, and we will
see that both equivalence classes and partitions are two sides of the same coin. It is a common
occurrence for us to have a set, and then to consider certain subsets of that set to represent a single
element, or to make all the elements in a subset the same.

There are lots of natural examples of equivalence relations.

Example. The rational numbers are the first place you see an equivalence relation, except it's so
natural that it's often swept under the rug and we don't even notice! A first attempt at defining
the set of rational numbers would be to consider the set

$$\left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \right\}.$$

This looks right, except for the fact that as written, the set thinks that $\frac{2}{4}$ and $\frac{3}{6}$ are different
elements. We want them to be the same! To deal with this we say

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{if} \quad ad = bc.$$

Now we can correctly write

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \right\} / \sim$$

which is just fancy notation for "the set $\left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \right\}$ except when $\frac{a}{b} \sim \frac{c}{d}$, we just make them
equal". The relation $\sim$ which makes two different things equal is called an **equivalence relation**.

Example. One that you have seen in this course is an equivalence relation on the integers $\mathbb{Z}$.
Consider the set of numbers that have the same remainder when divided by 7. Then we can say
that two integers $a, b \in \mathbb{Z}$ are equivalent if $7 \mid a - b$. Then this is an equivalence relation (see
Proposition 25), and it splits the integers up into 7 disjoint sets, one corresponding to each element
of $\mathbb{Z}_7$. In this scenario, we view all the integers which have the same remainder when divided by 7
as the same. Alternatively, we can view this as identifying all of these elements with each other.

Notice that the set of subsets consisting of elements of $\mathbb{Z}$ that are equivalent (which we will soon
call equivalence classes) cover all of $\mathbb{Z}$, and no element belongs to two of these subsets. Another
way of saying this is that the subsets **partition** $\mathbb{Z}$.

Example. Consider the set $X = \{1, 2, 3, 4, 5, 6\}$ and identify 1 and 2 together, and identify 3, 4,
and 6 as the same element. In this case, the equivalence relation identifies each of the subsets
$\{1, 2\}, \{3, 4, 6\}, \{5\}$. The idea is that we may as well just consider 1 to be equal to 2, and 3 to be
equal to 4 to be equal to 6.

Notice that the subsets that define which elements are equal partition the set X. If we denote
equality in this example by $\sim$ we also notice that for all $x, y, z \in X$, $x \sim x$, if $x \sim y$ then $y \sim x$,
and if $x \sim y$ and $y \sim z$, then $x \sim z$.

This last example hints at what we would formally want an equivalence relation to be, and we would like it to imitate what we understand "=" to mean. Intuitively in any context, $x = x$ always, if $x = y$ then $y = x$, and if $x = y$ and $y = z$, then $x = z$. Furthermore, if we group equal things together in a set, we should get a partition of that set into subsets consisting of things that are equal!

Let's formalise these ideas. In the next definition, think of a relation as something like $=$ or $\leq$. So something that takes in two elements of a set and gives back that it is either true or false.

Definition. Let X be a set. An **equivalence relation** on X is a relation $\sim$ that satisfies the following properties.

1. $x \sim x$ for all $x \in X$. We say $\sim$ is **reflexive**.

2. If $x \sim y$ then $y \sim x$. We say $\sim$ is **symmetric**.

3. If $x \sim y$ and $y \sim z$, then $x \sim z$. We say $\sim$ is **transitive**.

Definition. Let X be a set and $\sim$ an equivalence relation on X. For $x \in X$, define the **equivalence class of** x by

$$[x] := \{y \in X : y \sim x\}.$$

The set of equivalence classes is denoted by

$$X/\sim := \{[x] : x \in X\}.$$

Definition. Let X be a set. A **partition of** X is a collection of subsets $\{Y_i\}$ such that $Y_i \cap Y_j = \emptyset$ if $i \neq j$, and $\bigcup_i Y_i = X$.

Intuitively a partition is a way to split up your set into a collection of subsets in such a way that every element of X belongs to exactly one of the subsets.

As we have seen above, we can interchange the idea of an equivalence relation on a set with the notion of a partition. Intuitively they come hand in hand, and this is what the next proposition shows.

Proposition 77. *Let X be a set. If $\sim$ is an equivalence relation on X, then the set of equivalence classes of elements in X partition X. Conversely, if we have a partition of X, it arises as the set of equivalnce classes from an equivalence relation on X.*

Proof. Suppose $\sim$ is an equivalence relation on X. Since $x \in [x]$, every element is in an equivalence class. It remains to show that two equivalence classes are disjoint or equal. Suppose $y \in [x_1] \cap [x_2]$, we want to show that $[x_1] = [x_2]$. Let $z \in [x_1]$. Then since $y \in [x_1]$, $z \sim y$ and since $y \in [x_2]$, $y \sim x_2$. By transitivity of $\sim$, $z \sim x_2$ so $z \in [x_2]$. Conversely, if $z \in [x_2]$, $z \sim y$ and $y \sim x_1$ so $z \sim x_1$ and $z \in [x_1]$. Therefore $[x_1] = [x_2]$.

For each equivalence class $\mathcal{T} \in X/\sim$, pick an element $x \in \mathcal{T}$ and define the subset $Y_x \subset X$ by $Y_x = [x]$. Then by the discussion in the previous paragraph, $\bigcup Y_x = X$ and $Y_x \cap Y_y = \emptyset$ if $x \neq y$. Therefore the set of equivalence classes partitions the set X.

Conversely, suppose $\{Y_i\}$ is a partition of X. Then define a relation on X by $x \sim y$ if and only if $x \in Y_i$ and $y \in Y_i$ for the same subset Y_i. Alternatively, the relation is defined by $x \sim y$ if they both belong to the same subset Y_i. Since if $x \in Y_i$, then $x \in Y_i$, we have $x \sim x$. If x and y are both in Y_i for some i, then y and x both belong to the same subset, so $x \sim y$ implies $y \sim x$. Finally, if $x \sim y$ and $y \sim z$ we must have that $x, y \in Y_i$ and $y, z \in Y_j$ for some i, j. Then $y \in Y_i \cap Y_j$, and

since $\{Y_i\}$ is a partition, $Y_i = Y_j$. Therefore $x, z \in Y_i$ and $x \sim z$. This shows $\sim$ is an equivalence relation.

It remains to show the subsets Y_i are exactly the equivalence classes of $\sim$. Let $x \in X$, then $x \in Y_i$ for some i. Then $[x] = \{y \in X : y \sim x\} = \{y \in X : y \in Y_i\} = Y_i$, completing the proof. ■

The important take-home message from this proposition is that we can think of equivalence relations and partitions as two sides of the same coin. This theme occurs lots of times in this course, and it's always useful to have multiple ways of thinking about the same thing.

C   Some Set Theory

The goal of this appendix is to go over some facts about functions between sets, and what we can say about sizes of sets by looking at functions between sets. It is often the case that what is coming up is taken for granted. For example, at various points in the notes above we prove that a function is a bijection by finding an inverse. In this appendix we will show that these kinds of techniques actually work!

C.1   Injections, Surjections, and Bijections

Intuitively we know the definitions of an injection, surjection and bijection. An injection from S to T is a function that doesn't send any two elements of S to the same element of T. A surjection from S to T is a function that sends something to everything in T, or a function that hits everything in T. A bijection is a perfect matching, kind of like a dictionary, between elements of S and elements of T. That is, every element of S has an element of T associated to it, and vice versa. This is the same as saying that f is both surjective and injective. Let's make these intuitions formal.

Definition. Let $f : S \to T$ be a function.

- We say f is **injective** (or f is an **injection**) if whenever $f(s_1) = f(s_2)$, we have $s_1 = s_2$.

- We say f is **surjective** (or f is a **surjection**) if for all $t \in T$, there exists an $s \in S$ such that $f(s) = t$.

- We say f is **bijective** (or f is a **bijection**) if f is both injective and surjective.

This definition is all well and good, but there is another way to think about injections, surjections, and bijections. The idea is as follows.

If $f : S \to T$ is an injection, then every element in T that gets hit has a unique preimage (a unique element $s \in S$ such that $f(s) = t$) so we can define a $g : T \to S$ such that if we do f first and then g, we can return every element in S to itself.

If $f : S \to T$ is a surjection, then every $t \in T$ has at least one preimage, so we can define $g : T \to S$ to be a function that sends t to one of its preimages. Since every t has a preimage, this function has the property that if we do g first and then f, every element of t ends up back where it started.

If $f : S \to T$ is a bijection, then we can do what we did for the injections and surjections in a unique way to get a $g : T \to S$ such that $fg(t) = t$ for all $t \in T$ and $gf(s) = s$ for all $s \in S$.

These ideas are formalised in the following proposition. Here is a bit of notation we will use for the proposition and throughout the notes above. Let S be a set and define the identity function $\mathrm{id}_S : S \to S$ by $\mathrm{id}_S(s) = s$ for all $s \in S$.

Proposition 78. *Let* $f : S \to T$ *be a function between sets.*

- f *is an injection if and only if there exists a function* $g : T \to S$ *such that* $gf = \mathrm{id}_S$.

- f *is a surjection if and only if there exists a function* $g : T \to S$ *such that* $fg = \mathrm{id}_T$.

- f *is a bijection if and only if there exists a function* $g : T \to S$ *such that* $fg = \mathrm{id}_T$ *and* $gf = \mathrm{id}_S$. *Furthermore, such a* g *is unique and we denote it* $g = f^{-1}$.

Proof. Suppose f is an injection. Pick an $x \in S$ and for every $t \in f(S)$, let $s_t \in S$ be the unique element of S such that $f(s_t) = t$. Recall $f(S) := \{t \in T : \text{there exists } s \in S \text{ such that } f(s) = t\}$. Define $g : T \to S$ by

$$g(t) = \begin{cases} s_t & \text{if } t \in f(S) \\ x & \text{otherwise.} \end{cases}$$

Since every $s \in S$ is of the form s_t for some $t \in T$, we see $gf(s_t) = g(t) = s_t$ for all $s_t \in S$ so $gf = \mathrm{id}_S$.

Conversely suppose $f(a) = f(b) = t_0$ where $a \neq b$ in S. Suppose $g : T \to S$ is such that $gf = \mathrm{id}_S$. If $g(t_0) \neq a$, then $gf(a) \neq a$, so we must have $g(t_0) = a$. Then we have $gf(b) = a \neq b$, so such a g cannot exist.

Suppose f is a surjection. Define $g : T \to S$ by $g(t) = s_t$ where $f(s_t) = t$. Note that since f is surjective, we can always do this. Then $fg(t) = f(s_t) = t$ for all $t \in T$, so $fg = \mathrm{id}_T$.

Conversely, if f is a not a surjection there is some $t_0 \in T$ such that there is no $s \in S$ such that $f(s) = t_0$. Let $g : T \to S$ be a candidate function such that $fg = \mathrm{id}_T$. Then $fg(t_0) \neq t_0$ since there is no element in S such that $f(s) = t_0$. Therefore there is no function $g : T \to S$ such that $fg = \mathrm{id}_T$.

Finally, if f is a bijection, then define $g : T \to S$ to be $g(t) = s_t$ where $s_t \in S$ is the unique element such that $f(s_t) = t$. Note that every element in S is of the form s_t for some t. Then

$$gf(s_t) = g(t) = s_t \quad \text{and} \quad fg(t) = f(s_t) = t$$

for all $s_t \in S$ and $t \in T$, so $gf = \mathrm{id}_S$ and $fg = \mathrm{id}_T$. Conversely, if f is not injective or surjective, the same arguments above show that there cannot exist a $g : T \to S$ such that $fg = \mathrm{id}_S$ or $gf = \mathrm{id}_T$ respectively.

It remains to show that in the case when f is a bijection, the inverse g is unique. Suppose there is another map $h : T \to S$ such that $fh = \mathrm{id}_S$. Then $f(h(t)) = t = f(g(t))$ fur all $t \in T$. Since f is sinjective, we must have $h(t) = g(t)$ for all $t \in T$, completing the proof. ∎

As is discussed several times in the notes above, whenever you have a sutuation like this, you can use either property as the definition in your head. For example, we can now think of an injection has a map with a left inverse, or as a map which sends different elements to different elements. Whichever definition is easier or more helpful in a particular situation should be the one you use.

It is worth noting here that the above proof relies on the axiom of choice, but that is a discussion for another time and course.

In Lemma 32 we proved that multiplication by a unit induced a bijection on $\mathbb{Z}_n^*$. Let's prove it again here (this time for all of $\mathbb{Z}_n$, not just $\mathbb{Z}_n^*$), this time making use of Proposition 78

Lemma 79. *Let* $[a] \in \mathbb{Z}_n^*$. *The function* $\psi : \mathbb{Z}_n \to \mathbb{Z}_n$ *given by* $\psi([b]) = [a][b]$ *is a bijection.*

Proof. Consider the function $\theta : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $\theta([b]) = [a]^{-1}[b]$. Then $\theta\psi([b]) = \theta([a][b]) = [a]^{-1}[a][b] = [b]$ and $\psi\theta([b]) = \psi([a]^{-1}[b]) = [a][a]^{-1}[b] = [b]$. Therefore $\psi\theta = \mathrm{id}_{\mathbb{Z}_n}$ and $\theta\psi = \mathrm{id}_{\mathbb{Z}_n}$ so ψ is a bijection. ∎

C.2 Comparing the Sizes of Sets

There are lots of times you might want to show that two sets have the same size, or that one is bigger than the other. Here is a formal way to compare the size of two sets. Here, let $|S|$ be the size of a set.

Definition. Let S and T be sets.

- If there exists an injection $f : S \to T$, then $|S| \leq |T|$.

- If there exists a surjection $f : S \to T$, then $|S| \geq |T|$.

- If there exists a bijection $f : S \to T$, then $|S| = |T|$.

Recall that we used this definition in the proof of Lemma 39.

If $|S|$ and $|T|$ are finite, it is easy to see that this agrees with our intuition about what it means for a set to be bigger (or smaller) than another set. The advantage of this definition really becomes apparent when comparing infinite sets, because it gives us a formal way to say whether or not an infinite set is bigger or smaller than another infinite set.

When infinite sets get involved though, there are a few things that we need to check to make sure our notation using $\leq$ and $\geq$ actually makes sense. It turns out that these definitions work because of the following facts, which are far from obvious. They will be stated in terms of the existence of functions, and then in terms of what that means with regards to the relation $\leq$ defined above.

Fact 80. *Let S and T be sets.*

1. *There exists a surjection from S to T or from T to S (or both). Equivalently, $|S| \geq |T|$ or $|T| \geq |S|$ (or both).*

2. *There exists an injection $f : S \to T$ if and only if there exists a surjection $g : T \to S$. Equivalently $|S| \leq |T|$ if and only if $|T| \geq |S|$.*

3. *If there exists an injection $f_1 : S \to T$ and a surjection $f_2 : S \to T$, then there exists a bijection $f : S \to T$. Equivalently, if $|S| \geq |T|$ and $|S| \leq |T|$, then $|S| = |T|$.*

Facts 1 and 2 both rely on the axiom of choice, and Fact 3 is called the Schröder-Bernstein theorem.

With these definitions we can formally prove that $|\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{N}| = |\mathbb{Z} \times \mathbb{Z}|$. However, we have $|\mathbb{Z}| \lneq |\mathbb{R}|$, and we can prove that although there is an injection from $\mathbb{Z}$ to $\mathbb{R}$ (which is the regular inclusion map), we cannot find a bijection. If you're curious about this, look up Cantor's diagonalisation argument. It's one of the neatest lines of reasoning you'll see!

D Groups, rings, and fields

In this course, we have come across groups, rings, and fields, without formally talking about them. Let's do that to add a little algebraic context to our number theory.

D.1 Groups

The word group has popped up a bunch of times in the course, most notably in calling $\mathbb{Z}_n^*$ the group of units. What makes $\mathbb{Z}_n^*$ a group is that it comes with an operation (multiplication) that eats two elements of $\mathbb{Z}_n^*$ and spits out another element of $\mathbb{Z}_n^*$. More than that, there is a special element [1] (the identity with respect to the operation) with the property that $[1][a] = [a]$, and every element in $\mathbb{Z}_n^*$ has an inverse (with respect to the operation). Let's abstract these properties out into the definition of a group, which applies to many more structures in mathematics besides just $\mathbb{Z}_n^*$.

Definition. A **group** is a set G together with an operation $* : G \times G \to G$ such that the following properties are satisfied.

- For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.

- There exists an element e, such that for all $a \in G$, $e * a = a * e = a$ for all $a \in G$. We call e the **identity**.

- For every $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$. We call a^{-1} the **inverse** of a.

If we wish to emphases the operation, we will denote the group G by $(G, *)$.

Exercise. Consider $(\mathbb{Q} \setminus \{0\}, \div)$ where $\mathbb{Q} \setminus \{0\}$ is the set of non-zero rational numbers, and the operation $\div$ is just division, that is $a \div b = \frac{a}{b}$. Show that $(\mathbb{Q} \setminus \{0\}, \div)$ is not a group.

We have already come across a whole bunch of groups, as shown in the next exercise.

Exercise. Show that the following are groups, and verify the identity is indeed the identity.

- $(\mathbb{Z}_n^*, \cdot)$ (where $\cdot$ is multiplication in $\mathbb{Z}_n$), with identity [1].

- $(Q_n, \cdot)$ (where Q_n is the set of quadratic residues in $\mathbb{Z}_n^*$, and $\cdot$ is multiplication in $\mathbb{Z}_n$) with identity [1].

- $(\mathbb{Z}_n, +)$ with identity [0].

- $(\mathbb{Z}, +)$ with identity 0.

- $(\mathbb{R}_{>0}, \cdot)$ (where $\mathbb{R}_{>0}$ is the set of positive real numbers, and $\cdot$ is multiplication) with identity 1.

- $(\mathrm{SL}_2(\mathbb{R}), \cdot)$ (where $\mathrm{SL}_2(\mathbb{R})$ is the set of 2×2 matrices with real entries and determinant 1, and $\cdot$ is matrix multiplication) with identity $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

The first four groups in the exercise all have the property that $a * b = b * a$ for all a, b in the group, but the last one does not (for example $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$). All the groups we encounter in this course have this nice property, and they are given a special name.

Definition. Let $(G, *)$ be a group. If $a * b = b * a$ for all $a, b \in G$, then we say G is an **abelian group**.

So, $\mathbb{Z}_n^*$ (with multiplication) and $\mathbb{Z}_n$ (with addition) are abelian groups since $[a][b] = [b][a]$ and $[a] + [b] = [b] + [a]$ for all $[a], [b] \in \mathbb{Z}_n$. From here on in, if we write the group $\mathbb{Z}_n^*$, it is understood that the group operation is multiplication, since it's the only reasonable operation that turns $\mathbb{Z}_n^*$ into a group! Similarly for $\mathbb{Z}_n$ and addition. To reinforce this, we have the following exercise.

Exercise. Show that $\mathbb{Z}_n$ with multiplication is not a group. Show that $\mathbb{Z}_n^*$ with addition is not a group.

more to come...