Intersection Forms on Fermat Hypersurfaces in $\mathbb{CP}^3$

by

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under the supervision of

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1 Introduction

Fermat surfaces have long since been an object of geometric and arithmetic interest. Since the 15th century when Pierre de Fermat found columns too restricting to fully write down his truly marvellous proof of what would become known as Fermat's last theorem, people have been fascinated with answering the question of how many strictly positive integer points there are on the curve $x^n + y^n - z^n = 0$ for $n > 2$. We now know, thanks to Andrew Wiles, that the answer is indeed 0. Other than Fermat's last theorem, there has been an effort made to understand other Fermat surfaces; for example, we know that there are exactly 27 distinct lines on the Fermat cubic surface $x^3 + y^3 + z^3 + w^3 = 0$.

In this paper, we will be interested in the 4-dimensional manifolds given by the Fermat hypersurface

$$S_d := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : \sum_{i=0}^{3} z_i^d = 0\} \subset \mathbb{CP}^3.$$

We wish to understand some of the topological structure of this family of complex surfaces by explicitly writing down their intersection forms. The ultimate goal of this paper is to prove theorem 7.6, following the method in [GS99, theorem 1.3.8]. On the way there, we will take a scenic route to build up the required machinery and intuition, and we will be taking a few stops along the way to take in the view.

In section 2 we will give a short recap of some assumed knowledge in homology and cohomology theory, and we will set some conventions about notation.

Once these foundations have been set, section 3 will take two fundamental theorems in algebraic topology, the Poincaré Duality Theorem and the Universal Coefficient Theorem, and apply them to the specific case of closed, oriented 4-manifolds. We will see in lemma 3.5 that the only torsion in any of the homology and cohomology modules comes from $H_1(X;\mathbb{Z})$, so things become wonderful in the case that X is simply connected.

Section 4 will bring with it the definition of the intersection form for 4-manifolds. We will build up the theory of symmetric bilinear forms over finitely generated $\mathbb{Z}$-modules, with the goal of the section to prove the classification theorem for indefinite unimodular forms, lemma 4.11. Along the way, lemma 4.3 will show that every intersection form Q_X on a closed 4-manifold X is unimodular.

We will then make an excursion in section 5 to the land of intersection theory. This section is intended to build intuition in thinking of the cup product as dual to the geometric intersection of submanifolds, and we will define the intersection product on homology to do so. In this section we will also define the Euler class of a vector bundle and show that the name is not purely coincidental, exposing in lemma 5.2 the close link between the Euler class of the tangent bundle of a manifold and its Euler characteristic.

The Chern, Pontrijagin and Stiefel-Whitney classes will be defined in section 6, completing the array of characteristic classes needed to perform the required calculations in the final section. We will calculate the total Chern class of $\mathbb{CP}^n$ in lemma 6.1 and will also explore the hyperplane bundle $\mathcal{O}(1)$, writing down its first Chern class as the Poincaré dual of a hyperplane in proposition 6.2. Lemma 6.6 and corollary 6.9 are the main results of this section, giving us the link between the characteristic classes of the tangent bundle of a 4-manifold X, and the rank, signature and parity of the intersection form Q_X.

Finally, section 7 will bring the paper to a conclusion, bringing together all the pieces and machinery that have been built up so far. We will prove the main theorem, theorem 7.6, giving us an explicit description of the intersection forms for S_d.

Of course, this project would not have been possible without the help, guidance and encouragement from a large group of people. First and foremost, acknowledgements must go to the plethora of mathematicians who laid the foundations for the material in this paper. In particular, thanks must go to Robert Gompf and András Stipsicz for providing the statement and proof of the main theorem of this paper in [GS99]. I am extremely grateful to John Milnor for his inspiring and enlightening expositions, along with James Stasheff and Dale Husemoller in [Mil63], [MS74] and [MH73]. Thanks must also go to Alexandru Scorpan, who provides an intuitive and entertaining exposition of 4-manifolds in [Sco05], and also to Marvin Greenberg and John Harper, who provided the reference [GH81] from which I obtained the vast majority of my knowledge of singular homology and cohomology. A massive thank you goes to the authors of the classic books, Glen Bredon ([Bre93]), Raoul Bott and Loring Tu ([BT82]), Phillip Griffiths and Joseph Harris ([GH78]), Allen Hatcher ([Hat02]), John Lee ([Lee03]), and Igor Shafarevich ([Sha94]), for their ever reliable and comprehensive references.

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2 Notation, Conventions and Assumptions

For this paper, a manifold will always be connected and a closed manifold is one which is compact and has no boundary. We will be assuming that the reader is comfortable with real and complex vector bundles, especially the tangent bundle of a manifold, the pullback of a bundle and the Whitney sum of two bundles. If you are not, [Lee03, §4-6] and [MS74, §2, §3] are both fantastic for learning about vector bundles. We will also be assuming a reasonable grasp on singular homology and cohomology of a manifold. If you feel your grasp is unreasonable, both [Hat02] and [GH81] are excellent resources.

We know that for any orientable n-manifold M, the homology module $H_n(M, \partial M; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and we make the following standard definition.

Definition. An orientation on the manifold is a choice of generator for this module, which we call the **fundamental class** and denote it $[M] \in H_n(M, \partial M; \mathbb{Z})$.

Note that complex manifolds are always orientable and have a canonical orientation coming from the biholomorphic maps patching the charts together.

The *Kronecker product* will be given by

$$\langle \ , \ \rangle : H^i(M; \mathbb{Z}) \times H_i(M; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

and the cup and cap products,

$$\begin{aligned} \cup : H^i(M; \mathbb{Z}) \times H^j(M; \mathbb{Z}) &\longrightarrow H^{i+j}(M; \mathbb{Z}) \quad \text{and} \\ \cap : H^i(M; \mathbb{Z}) \times H_j(M; \mathbb{Z}) &\longrightarrow H_{j-i}(M; \mathbb{Z}), \end{aligned}$$

will be written such that $\langle a \cup b, x \rangle = \langle a, b \cap x \rangle$.

Throughout the paper, whenever we are referring to a 4-dimensional manifold, it will be denoted X, and when we are referring to a general manifold, it will be denoted M. I promise to try my very hardest to stick to this convention.

3 Poincare Duality and the Universal Coefficient Theorem

In this section we will state the Poincaré Duality Theorem and the Universal Coefficient Theorem without proof, together with some of the important consequences, and apply them to the case of 4-manifolds. These two important theorems in algebraic topology will give us the required machinery to explicitly write down the relations between the homology and cohomology modules of compact, oriented 4-manifolds.

Theorem 3.1 (Poincaré Duality). *Let M be an oriented compact n-dimensional manifold. The map*

$$\begin{aligned} \cap[M] : H^k(M;\mathbb{Z}) &\longrightarrow H_{n-k}(M;\mathbb{Z}) \\ a &\longmapsto a \cap [M] \end{aligned}$$

is an isomorphism for all k. We will denote the inverse map by $PD : H_k(M;\mathbb{Z}) \to H^{n-k}(M;\mathbb{Z})$, that is for an element $x \in H_k(M;\mathbb{Z})$, $PD(x) \in H^{n-k}(M;\mathbb{Z})$ is such that $PD(x) \cap [M] = x$.

For non-compact manifolds, the same theorem applies except the isomorphism is given by

$$\cap[M] : H^k_c(M;\mathbb{Z}) \longrightarrow H_{n-k}(M;\mathbb{Z})$$

where $H^k_c(M;\mathbb{Z}) = \varinjlim_{K \text{ compact}} H^k(M, M\backslash K;\mathbb{Z})$ is the cohomology module with compact supports (see [GH81, p. 215] and [Hat02, p. 245]). For our purposes however, we are only concerned with the compact case. An important corollary of Poincaré duality and the fact that all compact manifolds can be embedded in Euclidean space is the following [GH81, p. 228].

Corollary 3.2. *The homology modules of a compact manifold are finitely generated.*

Because of this result, from here on in all modules in this paper will be assumed to be finitely generated. Now for the Universal Coefficient Theorem for cohomology [Hat02, p. 193].

Theorem 3.3 (Universal Coefficient Theorem). *Let G be a $\mathbb{Z}$-module. Then*

$$0 \longrightarrow \mathit{Ext}(H_{k-1}(M;\mathbb{Z}), G) \longrightarrow H^k(M;G) \longrightarrow \mathit{Hom}_{\mathbb{Z}}(H_k(M;G),\mathbb{Z}) \longrightarrow 0$$

is a split exact sequence, where Ext is the derived functor of the Hom functor. The surjection $\alpha : H^k(M;G) \longrightarrow \mathit{Hom}_{\mathbb{Z}}(H_k(M;G),\mathbb{Z})$ is given by $\alpha(a) = \langle a, \ \rangle$.

The construction of Ext will not be included here, but the important property of Ext is that $\mathrm{Ext}(H,\mathbb{Z})$ is isomorphic to the torsion submodule of H when H is a finitely generated $\mathbb{Z}$-modules [Hat02, p. 196]. We will be interested in the case where $G = \mathbb{Z}$ and before we focus on the 4-dimensional case, we will first prove the following algebraic lemma.

Lemma 3.4. *If H is a finitely generated $\mathbb{Z}$-module, then $\mathit{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$ is isomorphic to the free submodule of H.*

Proof. Write $H = F \oplus T$ where $F \cong \mathbb{Z}^{\oplus n}$, that is the direct sum of $\mathbb{Z}$ with itself n times, and T is the torsion submodule of H. First observe that if $t \in T$ and $a \in \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$, then $a(t) = 0$ since $mt = 0$ for some $m \in \mathbb{Z}_{>0}$ and thus $a(mt) = m(a(t)) = 0$. Now consider a module homomorphism

$$\phi : H \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$$

and first we will see that $T \subset \ker\phi$ for any homomorphism.

Let $t \in T$ so $mt = 0$ for some $m \in \mathbb{Z}_{>0}$ and assume $\phi(t) \neq 0 \in \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$. This means $\phi(t)(b) \neq 0$ for some $b \in H$, but $\phi(mt)(b) = \phi(0)(b) = 0 \Rightarrow m(\phi(t)(b)) = 0$ and since $m \neq 0$, $\phi(t)(b) = 0$, a contradiction. Therefore $T \subset \ker\phi$.

Now let $\{e_1, \dots, e_n\}$ be generators of each of the $\mathbb{Z}$ summands of F and let $e_i^* \in \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ be such that $e_i^*(e_j) = \delta_{ij}$. Define $\phi(e_i) := e_i^*$, $\phi(t) = 0$ for $t \in T$ and extend linearly. From this definition we have that $\ker \phi = T$. To see ϕ is surjective, consider an element $a \in \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ and since $a(t) = 0$ for any $t \in T$, we know a is entirely determined by how it acts on F, and therefore how it acts on e_i for all i. From this we see that $\{e_1^*, \dots, e_n^*\}$ generates all of $\mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ and we conclude that $\mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}) \cong H/T \cong F$. ∎

With this lemma out of the way, let us now turn out attention to the homology and cohomology modules of a closed, oriented 4-manifold X. We know that the homology modules are finitely generated, so write $H_i(X; \mathbb{Z}) \cong F_i \oplus T_i$ where $F_i \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{Z})$ is the free part and $T_i \cong \mathrm{Ext}(H_i(X; \mathbb{Z}), \mathbb{Z})$ is the torsion submodule. The following lemma is from [Sco05, p. 15].

Lemma 3.5. *Let X be a closed, oriented 4-dimensional manifold. The homology and cohomology modules are given as follows.*

$$\begin{array}{ll}
H_0(X;\mathbb{Z}) \cong \mathbb{Z} & H^0(X;\mathbb{Z}) \cong \mathbb{Z} \\
H_1(X;\mathbb{Z}) \cong F_1 \oplus T_1 & H^1(X;\mathbb{Z}) \cong F_1 \\
H_2(X;\mathbb{Z}) \cong F_2 \oplus T_1 & H^2(X;\mathbb{Z}) \cong F_2 \oplus T_1 \\
H_3(X;\mathbb{Z}) \cong F_1 & H^3(X;\mathbb{Z}) \cong F_1 \oplus T_1 \\
H_4(X;\mathbb{Z}) \cong \mathbb{Z} & H^4(X;\mathbb{Z}) \cong \mathbb{Z}.
\end{array}$$

Proof. Since X is connected, closed and oriented, $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_4(X, \partial X; \mathbb{Z}) = H_4(X; \mathbb{Z}) \cong \mathbb{Z}$, and by Poincaré duality, $H^4(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_4(X; \mathbb{Z}) \cong \mathbb{Z}$. Since the sequence in the universal coefficient theorem splits (theorem 3.3), we have

$$H^i(X; \mathbb{Z}) \cong \mathrm{Ext}(H_{i-1}(X; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(H_i(X; \mathbb{Z}), \mathbb{Z}) \cong T_{i-1} \oplus F_i.$$

Putting this together with Poincaré duality we get

$$\begin{aligned}
H^3(X;\mathbb{Z}) &\cong F_3 \oplus T_2 \cong F_1 \oplus T_1 \cong H_1(X;\mathbb{Z}) \\
H^2(X;\mathbb{Z}) &\cong F_2 \oplus T_1 \cong F_2 \oplus T_2 \cong H_2(X;\mathbb{Z}) \\
H^1(X;\mathbb{Z}) &\cong F_1 \oplus T_0 \cong F_3 \oplus T_3 \cong H_3(X;\mathbb{Z})
\end{aligned}$$

which gives us that $T_1 \cong T_2$, $T_3 \cong T_0 \cong \{0\}$ and $F_1 \cong F_3$ by comparing torsion and free submodules. We now have

$$\begin{aligned}
H_0(X;\mathbb{Z}) &\cong \mathbb{Z} \\
H_1(X;\mathbb{Z}) &\cong F_1 \oplus T_1 \\
H_2(X;\mathbb{Z}) &\cong F_2 \oplus T_1 \\
H_3(X;\mathbb{Z}) &\cong F_1 \\
H_4(X;\mathbb{Z}) &\cong \mathbb{Z}
\end{aligned}$$

and the cohomology modules follow from Poincaré duality. ∎

An important thing to notice here is that the only torsion anywhere to be seen comes from $H_1(X; \mathbb{Z})$. We know that $\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X; \mathbb{Z})$ so if $\pi_1(X) \cong \{0\}$, $T_1 \cong \{0\}$ and we have the following immediate corollary in the simply connected case.

Corollary 3.6. *Let X be a simply connected, closed, oriented 4-manifold. Then*

$$H_i(X; \mathbb{Z}) \cong H_{4-i}(X; \mathbb{Z}) \cong H^i(X; \mathbb{Z}) \cong H^{4-i}(X; \mathbb{Z})$$

for all i and all of the homology and cohomology modules are torsion free.

4 Intersection Forms

We now shift our attention to intersection forms on closed, orientable 4-manifolds. For a closed, orientable 4-manifold X, we can define a bilinear form on $H^2(X, \partial X; \mathbb{Z}) = H^2(X; \mathbb{Z})$ as follows.

Definition. Let X be a closed, orientable 4-manifold. Define the **intersection form** Q_X as

$$\begin{aligned} Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (a,b) &\longmapsto \langle a \cup b, [X] \rangle , \end{aligned}$$

where $[X] \in H_4(X;\mathbb{Z})$ is the fundamental class of X.

Note that since 2 is even, $a \cup b = b \cup a$ for all $a, b \in H^2(X, \partial X; \mathbb{Z})$, so $Q_X(a,b) = Q_X(b,a)$. Additionally, since the cup product and the Kronecker product are both bilinear, we have that Q_X is a symmetric bilinear form.

The intersection form can just as easily be defined for *compact*, orientable 4-manifolds on $H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z})$, which is how it is defined in [GS99, p. 7], but for our purposes, we will only be dealing with closed manifolds.

It is worth remarking that the name *intersection form* comes from the fact that if X is closed, then every class $\alpha \in H_2(X;\mathbb{Z})$ can be represented by an embedded surface Σ_α, that is $\alpha = i_*[\Sigma_\alpha]$ where $i : \Sigma_\alpha \to X$ is the inclusion map. Now consider $a, b \in H^2(X;\mathbb{Z})$ with poincare duals α, β and their embedded representative surfaces $\Sigma_\alpha, \Sigma_\beta$ respectively. If Σ_α and Σ_β are chosen so that they intersect transversally then they will intersect at distinct points since the codimension of both embeddings is 2 and thus the codimension of the intersection will be 4. $Q_X(a,b)$ is then equal to the number of points in $\Sigma_\alpha \cap \Sigma_\beta$ counted with sign. For a full treatment of this see [GS99, Prop. 1.2.3].

Now since X is compact, Poincaré duality tells us that $H^2(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z})$, so we can also define Q_X on $H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z})$ as

$$Q_X(c,d) := \langle PD(c) \cup PD(d), [X] \rangle$$

where $PD(c) \in H^2(X;\mathbb{Z})$ is the Poincaré dual of c as noted above. Note that if a or b has torsion, $Q_X(a,b) = 0$ since, without loss of generality, if $na = 0$ for some $n \in \mathbb{Z}_{>0}$,

$$nQ_X(a,b) = n \langle a \cup b, [X] \rangle = \langle na \cup b, [X] \rangle = \langle 0, [X] \rangle = 0$$

and since $n \neq 0$, $Q_X(a,b) = 0$. With this in mind, if we let T be the torsion submodule of $H_2(X;\mathbb{Z})$, we can consider Q_X on the free $\mathbb{Z}$-module $H_2(X;\mathbb{Z})/T$. We can now pick a basis for $H_2(X;\mathbb{Z})/T$, and if this free $\mathbb{Z}$-module has dimension r, we can represent Q_X on $H_2(X;\mathbb{Z})/T$ as a matrix $M \in M_{r \times r}(\mathbb{Z})$ with respect to the chosen basis.

At this point we will take a brief foray into the land of symmetric bilinear forms over finitely generated free $\mathbb{Z}$-modules and build up the theory a little before returning to the specific case of intersection forms. From here on in, all free $\mathbb{Z}$-modules are finitely generated. We first make the following definitions.

Definition. Let A be a free $\mathbb{Z}$-module with symmetric bilinear form $Q : A \times A \to \mathbb{Z}$. Pick a basis for A and let M be the matrix of Q with respect to that basis.

- The **rank** of Q, denoted $\mathrm{rk}(Q)$, is the dimension of A.
- Extend M over $A \otimes_{\mathbb{Z}} \mathbb{R}$ and diagonalise. Let b_2^+ be the number of $+1$ entries on the diagonal, and b_2^- the number of -1 entries.
- The **signature** of Q, denoted $\sigma(Q)$, is the difference $b_2^+ - b_2^-$.

- Q is **positive definite** if $\text{rk}(Q) = \sigma(Q)$, **negative definite** if $\text{rk}(Q) = -\sigma(Q)$, and **indefinite** otherwise.
- Q is **even** if $Q(\alpha, \alpha) \equiv 0 \mod 2$ for all $\alpha \in A$, and **odd** otherwise. The property of being even or odd is called the **parity** of Q.
- An element $x \in A$ is called **characteristic** if $Q(\alpha, x) \equiv Q(\alpha, \alpha) \mod 2$ for all $\alpha \in A$.
- The **determinant**, $\det Q$, is the determinant of M.
- Q is called **unimodular** if $\det Q = \pm 1$.

Remark. Observe that from these definitions we immediately get that if Q is even, then $0 \in A$ is characteristic.

For our particular purposes, we will see that we only care about unimodular forms (see lemma 4.3). In this case, the definitions tell us that for any unimodular form $\text{rk}(Q) = b_2^+ + b_2^-$ and we get that $\sigma(Q) \equiv \text{rk}(Q) \mod 2$. Also for a unimodular form Q, observe that being positive or negative definite is equivalent to $b_2^- = 0$ or $b_2^+ = 0$ respectively.

We will now deal with the potential issue of whether or not the determinant of Q is well defined, since it depends on our choice of basis.

Lemma 4.1. *The determinant of a bilinear form Q on a free $\mathbb{Z}$-module is well defined.*

Proof. Let $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ and $\mathcal{C} = \{\gamma_1, \ldots, \gamma_n\}$ be two different bases for a free rank n $\mathbb{Z}$-module A. Then

$$\beta_i = \sum_{j=1}^{n} d_{ji}\gamma_j \quad \text{and} \quad \gamma_i = \sum_{j=1}^{n} e_{ji}\beta_j$$

for $d_{ji}, e_{ji} \in \mathbb{Z}$. We see from this that $[d_{ij}], [e_{ij}] \in M_{n\times n}(\mathbb{Z})$ are change of bases matrices that are inverses of each other. Since a matrix $T \in M_{n\times n}(\mathbb{Z})$ is invertible if and only if $\det T = \pm 1$, it follows that any change of bases matrix on a free $\mathbb{Z}$-module has determinant ± 1. Now let M be the matrix for Q with respect to some basis. If P is a basis transformation to another basis, the matrix for Q with respect to that basis is $P^T M P$ and since $\det P = \pm 1$, $\det(P^T M P) = \det M$. Therefore $\det Q$ is well defined. ■

Notice that if A has rank n, $A \cong \mathbb{Z}^{\oplus n}$ and therefore it admits a standard basis $\mathcal{S} = \{e_1, \ldots, e_n\}$. It follows from the proof of the lemma that a subset $\mathcal{B} = \{a_1, \ldots, a_n\} \subset A$ is a basis if and only if when we write $a_i = \sum_{j=1}^{n} \alpha_{ji} e_j$, the matrix $[\alpha_{ij}]$ has determinant ± 1 since this is precisely the basis transformation from $\mathcal{B}$ to $\mathcal{S}$. This motivates the following definition.

Definition. Let Q_1, Q_2 be bilinear forms on free $\mathbb{Z}$-modules A_1, A_2 with matrices M_1, M_2 respectively. We say that Q_1 and Q_2 are **isomorphic** or **equivalent** if $M_1 = P^T M_2 P$ for some invertible integer matrix P. We denote this $Q_1 \cong Q_2$.

Observe that $P : A_1 \to A_2$ here can be viewed as a module isomorphism (since it is invertible) such that $Q_1(a, b) = Q_2(Pa, Pb)$. We now wish to make precise what we mean when we say kQ for some $k \in \mathbb{Z}$, since as we will see, this is *not* represented by the matrix kM.

Definition. If A_1, A_2 are free $\mathbb{Z}$-modules with bilinear forms Q_1, Q_2 respectively, the direct sum $Q = Q_1 \oplus Q_2$ on $A_1 \oplus A_2$ is the bilinear form represented by the matrix

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

where M_1 and M_2 are matrices representing Q_1 and Q_2 respectively. The matrix M is with respect to the basis formed by concatenating the bases of A_1 and A_2 in that order. With this in mind, we make the following definition of kQ for $k \in \mathbb{Z}$.

$$kQ := \begin{cases} \bigoplus_k Q & \text{for } k > 0, \\ |k|\,(-Q) & \text{for } k < 0, \\ 0 & \text{for } k = 0. \end{cases}$$

where $(-Q)(a, b) := -(Q(a, b))$ or alternatively if M is a matrix representing Q, then $-M$ represents $-Q$.

The following two lemmas provide plenty of incentive to focus our attention on unimodular forms.

Lemma 4.2. *A symmetric bilinear form Q on A is unimodular if and only if the map*

$$\begin{aligned} L : A &\longrightarrow Hom_{\mathbb{Z}}(A, \mathbb{Z}) \\ a &\longmapsto Q(a, \ \ ) \end{aligned}$$

is an isomorphism.

Proof. Fix a basis $\{a_1, \dots, a_n\}$ of A and let M be the matrix of Q with respect to this basis. Let $\{a_1^*, \dots, a_n^*\}$ be the dual basis, that is a basis such that $a_i^*(a_j) = \delta_{ij}$. Now

$$L(a_i) = \sum_{j=1}^{n} Q(a_j, a_i) a_j^*$$

which means the matrix with respect to this basis of L is given by $[b_{ij}]$ where $b_{ij} = Q(a_i, a_j)$, which is exactly the matrix M. Now L is an isomorphism if and only if M is invertible, which is the case if and only if $\det M = \pm 1$. Therefore L is an isomorphism if and only if Q is unimodular. ■

Lemma 4.3. *If X is a closed 4-manifold, then the intersection form Q_X is unimodular.*

Proof. By lemma 4.2 it suffices to show the map

$$\begin{aligned} L_X : H_2(X; \mathbb{Z})/T_2 &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(X; \mathbb{Z}), \mathbb{Z}) \\ a &\longmapsto Q_X(a, \ \ ) \end{aligned}$$

is an isomorphism where $T_2 \subset H_2(X; \mathbb{Z})$ is the torsion submodule. This will follow from Poincaré duality and the Universal Coefficient Theorem. Let $T^2 \subset H^2(X; \mathbb{Z})$ be the torsion submodule and consider the inverse Poincaré duality isomorphism

$$PD : H_2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathbb{Z}).$$

Since $T_2 \cong T^2$ from lemma 3.5, this isomorphism descends to

$$\begin{aligned} \widetilde{PD} : H_2(X; \mathbb{Z})/T_2 &\longrightarrow H^2(X; \mathbb{Z})/T^2 \\ [a] &\longmapsto [PD(a)], \end{aligned}$$

which is also an isomorphism. By abuse of notation, for the rest of the proof we will also refer to this map as PD. We now consider the isomorphism

$$\begin{aligned} \phi : H^2(X; \mathbb{Z})/\mathrm{Ext}(H_2(X; \mathbb{Z}), \mathbb{Z}) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(X; \mathbb{Z}), \mathbb{Z}) \\ a &\longmapsto \langle a, \ \ \rangle \end{aligned}$$

from the Universal Coefficient Theorem (theorem 3.3). Consider the composition $\varphi = \phi \circ PD$ and we get the isomorphism

$$\begin{aligned} \varphi : H_2(X;\mathbb{Z})/T_2 &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z}) \\ a &\longmapsto \langle PD(a), \ \rangle \,. \end{aligned}$$

It remains to show that $\varphi = L_X$. Let $a, b \in H_2(X;\mathbb{Z})$, then

$$\begin{aligned} L_X(a)(b) &= Q_X(a,b) \\ &= \langle PD(a) \cup PD(b), [X] \rangle \\ &= \langle PD(a), PD(b) \cap [X] \rangle \\ &= \langle PD(a), b \rangle \\ &= \varphi(a)(b) \end{aligned}$$

and thus $L_X = \varphi$, proving the lemma. ■

We now shift our effort to classifying all indefinite unimodular bilinear forms on free $\mathbb{Z}$-modules, with our goal being to prove lemma 4.11 below. To get there, we will need to use Serre's Classification Lemma (without proof) and the following lemma. For a full proof of Serre's Classification Lemma, see [MH73, ch. 2, theorem 5.3].

Fact 4.4. *Let Q_1, Q_2 be indefinite unimodular forms on A_1 and A_2 respectively. If they have the same rank, signature and parity, then they are equivalent.*

Lemma 4.5. *Let Q be a unimodular bilinear form on a free $\mathbb{Z}$-module A. If $x \in A$ is characteristic, then $Q(x,x) \equiv \sigma(Q) \mod 8$.*

Proof. First we will find a characteristic element on an indefinite form (note that Q may be positive or negative definite), so we can then apply the previous fact. Let

$$y = x + e_+ + e_- \in A' = A \oplus \mathbb{Z} \oplus \mathbb{Z}$$

where e_+ and e_- are generators for the first and second copies of $\mathbb{Z}$ respectively. Consider the bilinear form $Q' = Q \oplus \langle 1 \rangle \oplus \langle -1 \rangle$ on A'.

Claim 4.6. *The element $y \in A'$ is a charactersitic element of Q'.*

Proof. Consider an element $a' = a + a_+e_+ + a_-e_- \in A'$ where $a_+, a_- \in \mathbb{Z}$, $a \in A$. Then

$$Q'(a',a') = Q(a,a) + a_+^2 - a_-^2 \quad \text{and} \quad Q(y,a') = Q(x,a) + a_+ - a_-.$$

Now we know $Q(x,a) \equiv Q(a,a) \mod 2$ and we also have for any integer b that $b^2 \equiv b \mod 2$. Therefore, $Q'(y,a') \equiv Q'(a',a') \mod 2$ for all $a' \in A'$ and y indeed is characteristic. □

Now we have an indefinite unimodular form Q' since the summands $\langle 1 \rangle$ and $\langle -1 \rangle$ ensure $(b_2^+)' \geq 1$ and $(b_2^-)' \geq 1$. Now consider the form $P = (b_2^+ + 1)\langle 1 \rangle \oplus (b_2^- + 1)\langle -1 \rangle$ on A', we want to show that the rank, signature and parity of Q' and P agree. We see that

$$\mathrm{rk}(Q') = b_2^+ + 1 + b_2^- + 1 = \mathrm{rk}(P) \quad \text{and} \quad \sigma(Q') = b_2^+ + 1 - (b_2^- + 1) = \sigma(P).$$

For parity, notice that both forms are odd since $Q'(e_+, e_+) = 1$ and $P(e_1, e_1) = 1$ if e_1 is the first basis vector with respect to which P is determined. Since Q' and P have the same rank, signature and parity, we can apply fact 4.4 to conclude that

$$Q' = Q \oplus \langle 1 \rangle \oplus \langle -1 \rangle \cong (b_2^+ + 1)\langle 1 \rangle \oplus (b_2^- + 1)\langle -1 \rangle,$$

where Q' in this form is with respect to some basis $\{e_1, \ldots, e_N\}$ of A' and $N = \mathrm{rk}(Q')$.

Claim 4.7. *With respect to this basis, the characteristic element $y = \sum_{i=1}^{N} y_i e_i$ is such that y_i is odd for all i.*

Proof. First note that

$$Q'(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \text{ and } i \leq b_2^+ + 1, \\ -1 & \text{if } i = j \text{ and } i > b_2^+ + 1, \\ 0 & \text{if } i \neq j. \end{cases}$$

Since y is characteristic, $Q'(y, e_i) = y_i Q'(e_i, e_i) \equiv Q'(e_i, e_i) \mod 2$ for all i and we therefore get $y_i \equiv 1 \mod 2$ for all i. □

If we now consider $Q'(y, y)$ we see

$$Q'(y, y) = \sum_{i=1}^{N} y_i^2 Q'(e_i, e_i) = \sum_{i=1}^{b_2^+ + 1} y_i^2 - \sum_{i=b_2^+ + 2}^{N} y_i^2.$$

Since y_i is odd for all i and an odd number squared is congruent to 1 mod 8, we see $Q'(y, y) \equiv (b_2^+ + 1) - (b_2^- + 1) = \sigma(Q) \mod 8$. It remains to show that $Q(x, x) = Q'(y, y)$ which follows immediately since

$$Q'(y, y) = Q'(x + e_+ + e_-, x + e_+ + e_-) = Q(x, x) + 1 - 1.$$

Alas, we now conclude that $Q(x, x) \equiv \sigma(Q) \mod 8$. ■

Corollary 4.8. *If Q is even, then $\sigma(Q) \equiv 0 \mod 8$ since $0 \in A$ is characteristic.*

Now we will define two bilinear forms which will be the building blocks for the even forms and we will prove some basic properties about them. Let E_8 and H be given by the following matrices,

$$E_8 := \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad H := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Lemma 4.9. *The forms E_8 and H are both even unimodular forms such that $\sigma(E_8) = 8$ and $\sigma(H) = 0$.*

Proof. Taking determinants of both their representative matrices we see that $\det E_8 = 1$ and $\det H = -1$ so they are both unimodular. Diagonalising over $\mathbb{R}$ we see that E_8 has 8 positive eigenvalues, giving $\sigma(E_8) = 8$, and H has eigenvalues $+1$ and -1 giving $\sigma(H) = 0$. To see both forms are even, we prove the following claim.

Claim 4.10. *If a matrix $[a_{ij}]$ representing a symmetric bilinear form Q on A is such that a_{ii} is even for all i, then Q is an even form.*

Proof. Write $a_{ii} = 2\beta_{ii}$ for $\beta_{ii} \in \mathbb{Z}$ for all i. Let $\{e_1, \ldots, e_n\}$ be the basis with respect to which $[a_{ij}]$ is the matrix of Q and consider an element $x = \sum_{i=1}^{n} x_i e_i \in A$. Then

$$Q(x, x) = Q\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} x_i e_i\right) = 2\sum_{i=1}^{n} \beta_{ii} x_i^2 + 2 \sum_{\substack{i<j \\ 1 \leq i \leq n \\ 1 \leq j \leq n}} a_{ij} x_i x_j \equiv 0 \mod 2.$$ □

Applying the claim to E_8 and H we see that they are both even, proving the lemma. ■

Now we can prove the following result which classifies all indefinite unimodular forms.

Lemma 4.11. *Suppose that Q is an indefinite, unimodular form. If Q is odd, then it is isomorphic to $b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$. If Q is even, it is isomorphic to $\frac{\sigma(Q)}{8}E_8 \oplus \frac{rk(Q)-|\sigma(Q)|}{2}H$.*

Proof. Fact 4.4 tells us that it suffices to find an indefinite unimodular form for every possible signature and rank for both odd and even forms. Recall that $\sigma(Q) \equiv \mathrm{rk}(Q) \mod 2$ and since Q is unimodular we have $b_2^+ = \frac{1}{2}(\mathrm{rk}(Q)+\sigma(Q))$ and $b_2^- = \frac{1}{2}(\mathrm{rk}(Q)-\sigma(Q))$. Since our assumption is that Q is indefinite, we have the additional constraint that $\mathrm{rk}(Q) > |\sigma(Q)|$.

For the odd case, given any rank r and signature σ, we get b_2^+ and b_2^- and setting the form

$$Q = b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$$

gives an odd form of rank r and signature σ (it is odd since $Q(e,e) \equiv 1 \mod 2$ for any basis vector e).

For the even case, given rank r and signature σ we have the additional constraint that $\sigma \equiv 0 \mod 8$ (corollary 4.8). This implies that $r \equiv 0 \mod 2$ and we can build up an even form with rank r and signature σ using E_8 and H as follows. We can first get a form of the required signature by using $\frac{\sigma}{8}E_8$, since $\sigma(E_8) = 8$. Since $\sigma(H) = 0$, adding copies of H doesn't change the signature. Since r is even, we can obtain the required rank by adding the appropriate number of copies of H. We can now conclude that the form

$$Q = \frac{\sigma}{8}E_8 \oplus \frac{r-|\sigma|}{2}H$$

is an even form of rank r and signature σ, completing the proof of the lemma. ■

5 Intersection Theory and the Euler Class

In this section we will give a quick survey of intersection theory to build up some important tools, machinery and most importantly intuition for the proofs and definitions to come. This section will be light on the details and heavy on the results. For more details (and just as many results) see [Bre93, Ch. 6 §11, §12].

Definition. Let M be an n-dimensional manifold. Define the **intersection product**

$$\bullet : H_i(M;\mathbb{Z}) \times H_j(M;\mathbb{Z}) \longrightarrow H_{i+j-n}(M;\mathbb{Z})$$

by $PD(a \bullet b) := PD(b) \cup PD(a)$.

From this definition we get

$$a \bullet b = (-1)^{(n-i)(n-j)} b \bullet a \quad \text{and} \quad a \bullet (b \bullet c) = (a \bullet b) \bullet c.$$

This concept is very closely related to the intersection form of a four manifold X. Notice that if $i = j = 2$, then $a \bullet b \in H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. The integer to which $a \bullet b$ corresponds is exactly $Q_X(a,b)$, which we see since

$$Q_X(a,b) = \langle PD(a) \cup PD(b), [X]\rangle = \langle PD(a \bullet b), [X]\rangle = \langle 1, a \bullet b\rangle .$$

Notice that since $a, b \in H_2(X;\mathbb{Z})$, the order of the cup product and intersection product doesn't matter.

Now since this section is called intersection theory, we will hope to relate this to geometric intersections. Given an oriented m-dimensional manifold M and an oriented n-dimensional submanifold N, we can define the homology class $[N]$ as follows.

Definition. Let $i : N \to M$ be the inclusion map, and let $[N] \in H_n(N;\mathbb{Z})$ be the fundamental class of N. Then we can view the **class of** N **in** M as $i_*[N]$, which we simply denote $[N] \in H_n(M;\mathbb{Z})$.

This definition leads us to the following fact proved in [Bre93, p. 372].

Fact 5.1. *Let M be an oriented m-dimensional manifold and let N and K be oriented n-dimensional and k-dimensional submanifolds respectively such that N and K intersect transversally. Then $[K \cap N] = [N] \bullet [K] \in H_{n+k-m}(M;\mathbb{Z})$.*

From this we have $PD([K \cap N]) = PD([K]) \cup PD([N])$ and we see that the cup product is dual to the transverse intersection of submanifolds. This is an extremely helpful way to think about cup products. Now we can define the Euler class of a vector bundle.

Definition. Let E be a rank k oriented vector bundle over a n-dimensional oriented manifold M and let W be the total space. Let $i : M \to W$ be a smooth embedding (we can view this as identifying M with the zero section of E). Define the **Euler class** as

$$e(E) := i^*(PD(i_*[M])) \in H^k(M;\mathbb{Z}).$$

Note that in this definition, the map PD is the inverse of $\cap[W] : H^i(W;\mathbb{Z}) \to H_{n+k-i}(W;\mathbb{Z})$, which is why we insist that E is an oriented vector bundle. Ignoring for a moment the push-forward and pullback maps, this defines the Euler class as the Poincaré dual in W of the fundamental class of M.

Now we realise this as an intersection as follows. First notice that any bundle over M can be viewed as the normal bundle when M is embedded into the total space of the bundle under the inclusion taking it to the zero section as above. Furthermore, we know we can associate an open neighbourhood of M with the total space of the normal bundle, and if M is closed in the total space, this neighbourhood is homotopy equivalent to M (see [MS74, §11]).

Now let M be an n-dimensional manifold, and W be such a neighbourhood in the total space of a rank k vector bundle. We can define the self intersection class of M as $[M] \bullet [M] \in H_{n-k}(W;\mathbb{Z})$ if M is n-dimensional. Geometrically, this corresponds to deforming M smoothly within the neighbourhood such that it intersects itself transversally, and $[M] \bullet [M]$ is the class of this intersection. If we let $i : M \to W$ be the inclusion map we then have

$$\begin{aligned}
[M] \bullet [M] &= (PD(i_*[M]) \cup PD(i_*[M])) \cap [W] \\
&= PD(i_*[M]) \cap (i_*[M]) \\
&= i_*(i^* PD(i_*[M]) \cap [M]) \\
&= i_*(e(\nu M) \cap [M])
\end{aligned}$$

where νM is the normal bundle. Now since W is homotopy equivalent to M, we have that $i_* : H_\bullet(M;\mathbb{Z}) \to H_\bullet(W;\mathbb{Z})$ is an isomorphism so we can view $e(\nu M) \cap [M] \in H_{n-k}(M;\mathbb{Z})$. We now have that $e(\nu M) = PD([M] \bullet [M]) \in H_k(M;\mathbb{Z})$. Since we can view any bundle this way by embedding M in the total space, we can make the following equivalent, and much more useful, definition of the Euler class of a vector bundle over a manifold.

Definition. Let F be a real rank n vector bundle over a m-dimensional manifold M. Pick a section $s : M \to F$ which intersects transversally with the zero section, $s_0(p) = (p, 0)$ for all $p \in M$. Let Z be the preimage of the zero set of s, $Z := s^{-1}(0) \in M$. Since s was chosen to intersect the zero section transversally, Z is an $(m-n)$-dimensional subset of M and we can therefore consider $[Z] \in H_{m-n}(M;\mathbb{Z})$. Define the **Euler class**

$$e(F) := PD([Z]) \in H_n(M;\mathbb{Z}).$$

We will close this section by drawing the link between the Euler class and the Euler characteristic of a manifold. First we define the latter and finish with the lemma below, for which we will only provide an outline of the proof. Complete proofs can be found in [Bre93, p. 379] and [MS74, p. 130].

Definition. Given a n-dimensional manifold M, define the **Euler characteristic**

$$\chi(M) := \sum_{i=0}^{n} (-1)^i \beta_i$$

where β_i is the ith **Betti number** and is given by the rank of the ith homology module $H_i(M;\mathbb{Z})$.

Lemma 5.2. *The Euler characteristic $\chi(M)$ of a smooth, closed, oriented n-manifold M is given by the equality $\langle e(TM), [M]\rangle = \chi(M)$, where $[M] \in H_n(M;\mathbb{Z})$ is the fundamental class of M.*

Sketch of proof. The total space of a manifold can be realised as the normal bundle of M, embedded via the diagonal map, in $M \times M$ [MS74, p. 121]. Since we are now dealing with $M \times M$, we would like to use the Künneth isomorphism, $H^\bullet(X \times Y) \cong H^\bullet(X) \otimes H^\bullet(Y)$, so we change the coefficients of homology and cohomology to a field (say $\mathbb{Q}$). We now pick a basis $\{\alpha_i\}$ and a dual basis $\{\alpha_i^*\}$ of $H^\bullet(M;\mathbb{Q})$, that is $\langle a_i^* \cup \alpha_j, [M]\rangle = \delta_{ij}$.

From this we end up with

$$\langle e(TM), [M]\rangle = \sum_i (-1)^{\deg(\alpha_i)} \langle \alpha_i^* \cup \alpha_i, [M]\rangle = \sum_i (-1)^{\deg(\alpha_i)} = \chi(M).$$

Note that changing the coefficients to $\mathbb{Q}$ not only allows us to use the Künneth isomorphism, but also doesn't change the rank of the homology modules, it only removes torsion leaving the Euler characteristic unchanged. ■

6 Characteristic Classes and the Intersection Form

We have already discussed the Euler class, an important characteristic class, in the previous section. Here we will explore the Chern, Pontrijagin and Stiefel-Whitney classes and see how they relate to important characteristics of the intersection form of a 4-manifold. The characteristic classes of a vector bundle E over a manifold M are elements of $H^\bullet(M;\mathbb{Z})$ (or $H^\bullet(M;\mathbb{Z}/2)$ for the Stiefel-Whitney class). From now on, when the vector bundle in question is the tangent bundle, we will denote all characteristic classes differently. For example, the total Chern class of the tangent bundle of a manifold, TM, will be denoted $c(M) := c(TM)$, and similarly for the other characteristic classes.

We begin this section by exploring the Chern classes of a complex vector bundle. For any rank n complex vector bundle E over a manifold M with total space W, denote $E_\mathbb{R}$ as the underlying rank $2n$ real vector bundle. Consider the space

$$W_0 := \{(p, v) \in W : v \neq 0\},$$

that is W_0 is the total space of E without the zero section. Let $\pi_0 : W_0 \to M$ be the restriction of the projection map to W_0.

Put a Hermitian metric on the bundle (and thus on each fibre) and construct a complex vector bundle on W_0 as follows. Define the fibre above each point $(p, v) \in W_0$ as $\langle v\rangle^\perp$. This gives us a rank $n-1$ complex bundle on W_0, which we will denote E_0.

Now any rank k bundle posseses an exact sequence, the *Gysin sequence* [MS74, p. 143], given by

$$\cdots \longrightarrow H^{i-k}(M) \xrightarrow{\cup e} H^i(M) \xrightarrow{\pi_0^*} H^i(W_0) \longrightarrow H^{i-k+1}(M) \longrightarrow \cdots$$

where $\cup e$ is cup product with the Euler class $e(E_{\mathbb{R}})$. For $i < 2n-1$, we have that

$$H^{i-2n}(M) \cong H^{i-2n+1}(M) \cong \{0\}$$

so $\pi_0^* : H^i(M) \to H^i(W_0)$ is an isomorphism and $(\pi_0^*)^{-1}$ makes sense. We are now ready to define the Chern classes.

Definition. Let E be a rank n complex vector bundle over a manifold M. Define the **Chern classes** $c_i(E) \in H^{2i}(M;\mathbb{Z})$ inductively as follows.

- For $i > n$, set $c_i(E) = 0$.
- Define $c_n(E) := e(E_{\mathbb{R}})$, the Euler class of the underlying real bundle.
- For $i < n$, set $c_i(E) := (\pi_0^*)^{-1} c_i(E_0)$.

The formal sum $c(E) := 1 + c_1(E) + \cdots + c_n(E) \in H^{\bullet}(M;\mathbb{Z})$ is called the **total Chern class** of E.

The Chern classes satisfy the following properties (see [MS74, §14] for a full treatment of this).

1. If $f : M \to M'$ is a smooth map between manifolds covered by a bundle map $E \to E'$ such that E and E' are of the same rank, then $c(E) = f^* c(E')$.
2. For a Whitney sum of vector bundles $E \oplus F$ over M, the *Whitney product formula* $c(E \oplus F) = c(E)c(F)$ holds.
3. If $\overline{E}$ is the conjugate bundle of a complex vector bundle E, then $c_i(\overline{E}) = (-1)^i c_i(E)$.
4. If E is the trivial bundle, then the total Chern class $c(E)$ is 1.

Now we define the Chern numbers.

Definition. Let M be a complex n-dimensional manifold and let $[M] \in H_{2n}(M;\mathbb{Z})$ be the fundamental class. Let $\{i_1, \ldots, i_k\}$ be a partition of n, then $c_{i_1}(M) \cup \cdots \cup c_{i_k}(M) \in H^{2n}(M;\mathbb{Z})$ and therefore can be evaluated on $[M]$ to give $\langle c_{i_1}(M) \cup \cdots \cup c_{i_k}(M), [M] \rangle \in \mathbb{Z}$. This integer is the **Chern number** corresponding to the partition $\{i_1, \ldots, i_k\}$ and will be denoted $c_{i_1} c_{i_2} \ldots c_{i_k}[M]$.

In our case, we will be specifically interested in complex 2-manifolds, where the possible Chern numbers are $c_1^2[M] := c_1 c_1[M]$ and $c_2[M]$.

Using the facts and properties of Chern classes above, we will now explicitly calculate the Chern class of $\mathbb{CP}^n$, following the method in [MS74, p. 169].

Lemma 6.1. *The total Chern class of $\mathbb{CP}^n$ is given by $c(\mathbb{CP}^n) = (1+g)^{n+1}$ where g is given by $g = c_1(\mathcal{O}(1)) \in H^2(\mathbb{CP}^n;\mathbb{Z})$ and $\mathcal{O}(1)$ is the dual of the tautological line bundle over $\mathbb{CP}^n$.*

Proof. Let $\mathcal{O}(-1)$ be the tautological line bundle on $\mathbb{CP}^n$, that is at each point $p \in \mathbb{CP}^n$, the fibre is the line corresponding to p itself, call it L_p. View $\mathcal{O}(-1)$ as a subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, that is each fibre of $\mathcal{O}(-1)$ is a line through the origin of a fibre of the trivial bundle, and define a rank n bundle ω^n as follows. Put a Hermitian metric on the trivial bundle and define the fibre of ω^n at a point $p \in \mathbb{CP}^n$ to be $L_p^{\perp}$. This gives us a rank n complex bundle ω^n such that $\mathcal{O}(-1) \oplus \omega^n$ is the complex rank $(n+1)$ trivial bundle.

Consider the bundle given by $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \omega^n)$ and identify it with the tangent bundle $T\mathbb{CP}^n$ as follows. View S^{2n+1} embedded in $\mathbb{C}^{n+1}$ as $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$ and let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then we can view $\mathbb{CP}^n$ as the quotient, S^{2n+1}/S^1 and let $\pi : S^{2n+1} \to S^{2n+1}/S^1$ be the quotient map.

Now every point in the tangent bundle $T\mathbb{CP}^n$ can be associated with a set $\{(tx, tv) \in TS^{2n+1} : t \in S^1\}$ where $\langle x, x\rangle = 1$ and $\langle x, v\rangle = 0$. This is because under the map $\pi_* : TS^{2n+1} \to T\mathbb{CP}^n$, $\pi_*(x, v) = \pi_*(tx, tv)$ for all $t \in S^1$.

If we now fix a point $p \in \mathbb{CP}^n$, this is equivalent to fixing the unit vectors on the line L_p, which we will denote by $\{tx_p \in L_p : t \in S^1, |x_p| = 1\}$. An element of the fibre $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \omega^n)\big|_p$ is a linear map $\phi : L_p \to L_p^{\perp}$. Each element ϕ uniquely determines a set

$$\{(tx_p, t\phi(x_p)) \in T_pS^{2n+1} : t \in S^1\},$$

which determines an element of $T_p\mathbb{CP}^n$. Conversely, given an element of $T_p\mathbb{CP}^n$, we have an associated set $\{(tx_p, tv_p) \in T_pS^{2n+1}\}$ which uniquely determines an element of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \omega^n)\big|_p$ by setting $\phi(x_p) = v_p$. Under this association we have

$$T\mathbb{CP}^n \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \omega^n).$$

Now if we add the trivial line bundle $\mathbb{C} \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \mathcal{O}(-1))$ we get

$$\begin{aligned} T\mathbb{CP}^n \oplus \mathbb{C} &\cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \omega^n \oplus \mathcal{O}(-1)) \\ &\cong \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{O}(-1), \mathbb{C}^{\oplus(n+1)}\right) \qquad \text{since } \mathcal{O}(-1) \oplus \omega^n \text{ is trivial,} \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \mathbb{C})^{\oplus(n+1)}. \end{aligned}$$

If we define $\mathcal{O}(1) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \mathbb{C})$ we get $T\mathbb{CP}^n \oplus \mathbb{C} \cong \mathcal{O}(1)^{\oplus(n+1)}$. Using the Whitney product formula and the property that $c(\mathbb{C}) = 1$ we get

$$\begin{aligned} c(T\mathbb{CP}^n) &= c(T\mathbb{CP}^n \oplus \mathbb{C}) \\ &= c(\mathcal{O}(1)^{\oplus(n+1)}) \\ &= c(\mathcal{O}(1))^{n+1} \\ &= (1 + c_1(\mathcal{O}(1)))^{n+1}. \end{aligned}$$

Setting $g = c_1(\mathcal{O}(1))$ proves the lemma. ∎

Now we want to discuss what exactly the zero sets of global sections of $\mathcal{O}(1)$ look like as this will come in handy for calculating the Euler class later on. Global sections of $\mathcal{O}(1)$ can be constructed as follows. Consider a linear functional $f : \mathbb{C}^{n+1} \to \mathbb{C}$. For any line L through the origin, we get a linear functional on L by the restriction $f\big|_L : L \to \mathbb{C}$. Note that any linear functional is zero on an n-dimensional subspace, unless it is 0 everywhere.

Now given a point $p \in \mathbb{CP}^n$, the fibre $\mathcal{O}(1)_p$ is the space of linear functionals from the line L_p to $\mathbb{C}$, that is $\mathcal{O}(1)_p \cong \mathrm{Hom}_{\mathbb{C}}(L_p, \mathbb{C})$. A global section $s : \mathbb{CP}^n \to \mathcal{O}(1)$ is given by $s(p) = f\big|_{L_p}$ for some linear functional $f : \mathbb{C}^{n+1} \to \mathbb{C}$. The zero set of such a global section is exactly a hyperplane corresponding to the lines through the origin in $\mathbb{C}^{n+1}$ on which $f\big|_{L_p} = 0$.

Furthermore, if we are given a hyperplane H in $\mathbb{CP}^n$ given by $\sum_{i=0}^n a_i x_i = 0$ it is the zero set of a global section. To see this, pick a basis $\mathcal{B} = \{e_0, \dots, e_n\}$ of $\mathbb{C}^{n+1}$ corresponding to the coordinates $\{x_0, \dots, x_n\}$ and a Hermitian metric with respect to which $\mathcal{B}$ is orthogonal. Then all linear functionals on $\mathbb{C}^{n+1}$ can be written as $\langle v, \ \rangle$ for $v \in \mathbb{C}^{n+1}$. If we set $v = \sum_{i=0}^n a_i e_i$ then the global section given by restricting $f = \langle v, \ \rangle$ to each line L_p has a zero set which is exactly the hyperplane H. This is why $\mathcal{O}(1)$ is also called the *hyperplane bundle*. From this discussion and the definition of the Euler class above, we immediately get the following proposition.

Proposition 6.2. *The Euler class and first Chern class of $\mathcal{O}(1)$ over $\mathbb{CP}^n$ is given by*

$$e(\mathcal{O}(1)) = c_1(\mathcal{O}(1)) = PD([H])$$

for any hyperplane H in $\mathbb{CP}^n$.

It is worth remarking here that the notation of $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ is standard for the particular corresponding line bundles in the discussion above.

We now define another set of characteristic classes called the Pontrijagin classes. These are classes on real vector bundles and are defined using the complexification of a real vector bundle. Before defining them we will briefly discuss what the complexification looks like on each fibre.

Definition. Given a real vector space V, the **complexification** is a complex vector space given by $V \otimes \mathbb{C}$.

Every element in the complexification of a vector space $V \otimes \mathbb{C}$ can be written as $x + iy$ for some $x, y \in V$ and as a result, $V \otimes \mathbb{C} \cong V \oplus V$ as real vector spaces. Applying this to every fibre in a real rank n vector bundle F we get the complexification $F \otimes \mathbb{C}$, which is a complex rank n vector bundle isomorphic (as a real vector bundle) to the Whitney sum $F \oplus F$. We can now define the Pontrijagin classes as follows.

Definition. Given a real rank n vector bundle F over a manifold M, define the i**th Pontrijagin class** of F as $p_i(F) \in H^{4i}(M; \mathbb{Z})$ given by

$$p_i(F) := (-1)^i c_{2i}(F \otimes \mathbb{C}).$$

Define the **total Pontrijagin class** $p(F) := 1 + p_1(F) + \cdots + P_{\lfloor \frac{n}{2} \rfloor}(F)$ where $\lfloor \frac{n}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n}{2}$.

Now we want to express these Pontrijagin classes in terms of the Chern classes, with the intention of proving lemma 6.6 below. To this this, we first need the following result.

Lemma 6.3. *For any complex vector bundle E, the vector bundle $E \otimes \mathbb{C}$ is isomorphic to the Whitney sum $E \oplus \overline{E}$ where $\overline{E}$ is the conjugate bundle.*

Proof. We will construct an isomorphism on each fibre individually, which will extend to an isomorphism of vector bundles. Given a vector space V, we know that any element in $V \otimes \mathbb{C}$ can be written as $x + iy$ for $x, y \in V$. Now let V be a complex vector space and consider the map

$$\begin{aligned} \phi(V \otimes \mathbb{C}) &\longrightarrow V \oplus \overline{V} \\ x + iy &\longmapsto \left(\frac{1}{2}(x + iy), \frac{1}{2}(x - iy)\right). \end{aligned}$$

Given $\alpha \in \mathbb{C}$ we have

$$\phi(\alpha(x + iy)) = \left(\alpha\frac{1}{2}(x + iy), \overline{\alpha}\frac{1}{2}(x - iy)\right) = \alpha(\phi(x + iy))$$

and since ϕ clearly commutes with addition we have a complex linear map. Now we see that if $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{(v_1, iv_1), \ldots, (v_n, iv_n), (v_1, -iv_1), \ldots, (v_n, -iv_n)\}$ is a basis for $V \oplus \overline{V}$ so $\dim_{\mathbb{C}}(V \oplus \overline{V}) = 2n = \dim_{\mathbb{C}}(V \otimes \mathbb{C})$. Lastly, note $\ker \phi = 0$ and therefore $V \otimes \mathbb{C} \cong V \oplus \overline{V}$ as complex vector spaces. Since this map is natural and independent of any choice of basis, we can extend this isomorphism to the vector bundle isomorphism $E \otimes \mathbb{C} \cong E \oplus \overline{E}$ for a complex vector bundle E. ∎

With this result we can now write down an expression for the Pontrijagin classes in terms of the Chern classes.

Lemma 6.4. *For any complex rank n vector bundle E over a manifold M, the relation*

$$1 - p_1(E) + p_2(E) - \cdots + (-1)^n p_n(E) = (1 + c_1(E) + \cdots + c_n(E))(1 - c_1(E) + \cdots + (-1)^n c_n(E))$$

holds. In particular we have $p_1(E) = c_1^2(E) - 2c_2(E)$.

Proof. From the Whitney product formula we have $c(E \otimes \mathbb{C}) = c(E \oplus \overline{E}) = c(E)c(\overline{E})$, and we know $c_i(E) = (-1)^i(\overline{E})$ so

$$c(E \otimes \mathbb{C}) = c(E)c(\overline{E}) = (1 + c_1(E) + \cdots + c_n(E))(1 - c_1(E) + \cdots + (-1)^n c_n(E)).$$

Once this is expanded, all odd Chern classes vanish and comparing dimensions we are left precisely with the required formula. Note that the alternating sum on the left hand side is from the factor of $(-1)^i$ in the definition, $p_i(E) := (-1)^i c_{2i}(E \otimes \mathbb{C})$. In particular, for the first Pontrijagin class, looking at the terms in $H^{4i}(M)$ we get

$$c(E \otimes \mathbb{C}) = (1 + c_1(E) + c_2(E) + \cdots)(1 - c_1(E) + c_2(E) + \cdots) = 1 + (2c_2(E) - c_1^2(E)) + \cdots$$

and thus $p_1(E) = c_1^2(E) - 2c_2(E)$. ∎

We are now one great big hammer away from having the machinery to prove lemma 6.6, and that hammer is the following fact, which is the 4-dimensional instance of the Hirzebruch Signature Theorem. A proof of the general statement can be found in [MS74, p. 224].

Fact 6.5 (Hirzebruch Signature Theorem for 4-Manifolds)**.** *If X is a smooth, closed, oriented 4-manifold, then $\sigma(Q_X) = \frac{1}{3}\langle p_1(X), [X]\rangle$.*

Lemma 6.6. *Let X be a simply connected, smooth, closed, oriented 4-manifold with intersection form Q_X. Then*

$$c_2[X] = \chi(X) = 2 + rk(Q_X) \quad \text{and} \quad c_1^2[X] = 3\sigma(Q_X) + 2\chi(X).$$

Proof. For the first equality, we know $c_2(X) = e(X)$ and thus $c_2[X] = \langle e(X), [X]\rangle = \chi(X)$ by lemma 5.2. However, $\chi(X) = \sum_{i=0}^4 (-1)^i \beta_i$ where β_i is the rank of $H_i(X;\mathbb{Z})$. Now Poincaré duality tells us that $\beta_i = \beta_{4-i}$ for all i. Since X is simply connected, $H_1(X;\mathbb{Z}) = \{0\}$ and $\beta_1 = \beta_3 = 0$ and also the connectedness implies $\beta_0 = \beta_4 = 1$. From the definition of the rank of a quadratic form we know that $\mathrm{rk}(Q_X)$ is the rank of the free module it acts on, which in this case is the rank of $H_2(X;\mathbb{Z})$, or β_2. Putting all of this together we get

$$c_2[X] = \chi(X) = \sum_{i=0}^{4} \beta_i = 2 + \mathrm{rk}(Q_X).$$

The second equality follows from fact 6.5 above as follows. Lemma 6.4 gives us

$$\langle p_1(X), [X]\rangle = \left\langle c_1^2(X) - 2c_2(X), [X]\right\rangle = c_1^2[X] - 2c_2[X].$$

The fact above now gives $\sigma(Q_X) = \frac{1}{3}(c_1^2[X] - 2c_2[X])$ and since $c_2[X] = \chi(X)$ we have

$$c_1^2[X] = 3\sigma(Q_X) + 2\chi(X)$$

completing the proof. ∎

Before moving on to the next section, we will mention the Stiefel-Whitney classes, which are characteristic classes $w_i(E) \in H^i(M;\mathbb{Z}/2)$. Historically, these classes came first but they play a very small part in this exposition. We will state some of their properties without proof and prove a result which will be needed later. For a more comprehensive and informative treatment, see [MS74, §4 and §8] and [Bre93, Ch 6, §17]. We will define them here axiomatically, the same way as is done in [MS74].

Definition. Let E be a real rank n vector bundle over a manifold M. There are a sequence of cohomology classes $w_i(E) \in H^i(M;\mathbb{Z}/2)$ called the **Stiefel-Whitney classes** that satisfy the following four axioms.

1. The class $w_0(E) = 1$ and $w_i(E) = 0$ if $i > n$.
2. If N is another manifold with vector bundle F and if $f : M \to N$ is covered by a bundle map $g : E \to F$, then $w_i(E) = f^* w_i(F)$.
3. If E and F are both bundles over M, then $w_k(E \oplus F) = \sum_{i=0}^{k} w_i(E) \cup w_{k-i}(F)$.
4. Let γ be the tautological line bundle over $\mathbb{RP}^1$. Then $w_1(\gamma) \neq 0$.

The element $w = 1 + w_1(E) + \cdots + w_n(E) \in H^\bullet(M; \mathbb{Z}/2)$ is called the **total Stiefel-Whitney class**.

It turns out that such classes always exist and can be explicitly constructed using the Thom isomorphism and Steenrod squares [Bre93, p. 421], [MS74, p. 91]. From this construction, the following two properties follow.

Fact 6.7. *Let E be a rank n vector bundle over M. Then*

1. *E is orientable if and only if $w_1(E) = 0$, and*
2. *Under the natural projection map $H^n(M; \mathbb{Z}) \to H^n(M; \mathbb{Z}/2)$, the Euler class $e(E)$ maps to the top Stiefel-Whitney class, $w_n(E)$.*

By abuse of notation, we write the second fact $e(E) \equiv w_n(E) \mod 2$. We now prove the following important lemma.

Lemma 6.8. *Let X be an oriented 4-manifold and $\alpha \in H_2(X; \mathbb{Z})$. Then*

$$\langle w_2(X), \alpha \rangle \equiv Q_X(\alpha, \alpha) \mod 2$$

where in the expression $\langle w_2(X), \alpha \rangle$, α is reduced to $\mathbb{Z}/2$ coefficients.

Proof. Represent $\alpha \in H_2(X; \mathbb{Z})$ by an embedded orientable surface Σ, that is $\alpha = i_*[\Sigma]$ where $i : \Sigma \to X$ is the inclusion map. Letting $\nu\Sigma$ be the normal bundle of Σ in X we have

$$\begin{aligned}
\langle w_2(X), \alpha \rangle &= \langle w_2(X), i_*[\Sigma] \rangle \\
&= \langle i^* w_2(X), [\Sigma] \rangle \\
&= \langle w_2(i^* TX), [\Sigma] \rangle \\
&= \left\langle w_2(TX\big|_\Sigma), [\Sigma] \right\rangle \\
&= \langle w_2(T\Sigma \oplus \nu\Sigma), [\Sigma] \rangle \\
&= \langle w_2(T\Sigma), [\Sigma] \rangle + \langle w_2(\nu\Sigma), [\Sigma] \rangle + \langle w_1(T\Sigma) \cup w_1(\nu\Sigma), [\Sigma] \rangle \\
&\equiv \chi(\Sigma) + \langle e(\nu\Sigma), [\Sigma] \rangle \mod 2,
\end{aligned}$$

since $w_1(T\Sigma) = 0$ and $e(\Sigma) \equiv w_2(\Sigma) \mod 2$. Now since Σ is orientable, a classical result tells us $\chi(\Sigma) = 2 - 2g \equiv 0 \mod 2$ where g is the genus of the surface. We know from intersection theory that the Euler class of the normal bundle when viewed in the ambient space $[X]$ is given by $PD([\Sigma] \bullet [\Sigma])$. This gives us

$$\langle w_2(X), \alpha \rangle \equiv \langle e(\nu\Sigma), [\Sigma] \rangle = \langle PD(i_*[\Sigma]) \cup PD(i_*[\Sigma]), [X] \rangle = Q_X(\alpha, \alpha) \mod 2,$$

completing the proof. ∎

From corollary 3.6 we know that if X is closed and simply connected then $H^2(X; \mathbb{Z})$ has no torsion. This gives us the following corollary.

Corollary 6.9. *Let X be a simply connected closed 4-manifold. The intersection form Q_X is even if and only if $w_2(X) = 0$.*

7 Fermat Surfaces

Consider the Fermat surface in $\mathbb{CP}^3$,

$$S_d := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 : \sum_{i=0}^{3} z_i^d = 0\} \subset \mathbb{CP}^3$$

where d is a positive integer.

Our goal is to prove theorem 7.6, and our first step is to prove $\pi_1(S_d) \cong \{1\}$, that is, S_d is simply connected. To do this, we will use the Lefschetz Hyperplane Theorem without proof (see [Mil63, §7] for a full treatment of this).

Theorem 7.1 (Lefschetz Hyperplane Theorem)**.** *Let X be a compact complex n-dimensional submanifold of $\mathbb{CP}^N$. If H is a hyperplane in $\mathbb{CP}^N$, then the homomorphism*

$$\pi_i(X \cap H) \longrightarrow \pi_i(X)$$

is an isomorphism for $i < n-1$ and a surjection for $i = n-1$.

With this ammunition, we can now prove the following lemma.

Lemma 7.2. *The 4-manifold S_d is simply connected.*

Proof. Consider the dth Veronese embedding

$$\nu_d : \mathbb{CP}^3 \longrightarrow \mathbb{CP}^N$$

where $N = \binom{d+3}{d} - 1$ is one less than the number of monomials of degree d in the four variables z_0, z_1, z_2, z_3. Let the map be given by

$$\nu_d([z_0 : z_1 : z_2 : z_3]) = [z_0^d : z_1^d : z_2^d : z_3^d : m_1 : \cdots : m_{N-3}]$$

where $\{m_i\}_{i=1}^{N-3}$ are all other monomials of degree d. Give $\mathbb{CP}^N$ coordinates $[X_0 : \cdots : X_N]$. The hyperplane H given by $X_0 + X_1 + X_2 + X_3 = 0$ contains the image of S_d under the Veronese embedding and if we restrict this hyperplane to the image of $\mathbb{CP}^3$ we exactly get $H \cap \nu_d(\mathbb{CP}^3) = \nu_d(S_d)$.

Since the Veronese embedding is an isomorphic embedding [Sha94, p. 52], we have that $\nu_d(\mathbb{CP}^3)$ is an embedded complex 3-manifold in $\mathbb{CP}^N$ and applying the Lefschetz Hyperplane Theorem we get $\pi_1(\nu_d(S_d)) \cong \pi_1(\nu_d(\mathbb{CP}^3))$. Since $\mathbb{CP}^3$ is simply connected we get $\pi_1(S_d) \cong \pi_1(\mathbb{CP}^3) \cong \{1\}$ and S_d is simply connected. ∎

We are now two short lemmas away from the main theorem of this exposition. From here on in, let $i : S_d \to \mathbb{CP}^3$ be the inclusion map.

Lemma 7.3. *The first Chern class of the normal bundle νS_d in $\mathbb{CP}^3$ is equal to dx where $x = i^*(g)$ and $g = c_1(\mathcal{O}(1))$ (see lemma 6.1).*

Proof. For this argument, first endow $\mathbb{CP}^n$ with a metric (for example, the Fubini-Study metric will do). Let $S_d' \subset \mathbb{CP}^3$ be another hypersurface cut out by a homogeneous polynomial of degree d. By altering the coefficients of the polynomial defining S_d as little or as much as need be, we can choose S_d' to be smooth, arbitrarily uniformly close to S_d and to intersect S_d transversally (full justification of this fact will be left out, but the smoothness is a consequence of Sard's theorem and the arbitrary closeness and transversality are consequences of Bertini's theorem). Since we have this freedom in the choice of S_d' we can choose it to be a section of the normal bundle νS_d, and the *general idea* of how we do this is as follows.

Note that the normal bundle νS_d is a line bundle since S_d is of codimension 1 in $\mathbb{CP}^3$. For a point $p \in S_d$, consider the normal line N_p passing through p. By Bezout's theorem [Sha94, §2.1], this line intersects S_d at at least one other point since N_p intersects S_d transversally at p for all p. Take the minimum distance between p and any of the other points and call this distance m_p. First note that since S_d is smooth, $m_p > 0$ for all p. Now let $\mu = \min_{p \in S_d}\{m_p\}$, which is attained since S_d is compact. Let $\varepsilon = \frac{1}{4}\mu$ and choose S'_d to be such that the distance between the two compact sets S_d and S'_d is less than ε and for every $p \in S_d$ there is exactly one point $p' \in S'_d$ such that $p' \in S'_d \cap N_p$ and $\operatorname{dist}(p, p') < \varepsilon$. Now for each point $p \in S_d$, choose this p' in the normal fibre N_p, defining the global section. This section will be S'_d.

If we now let $V = S_d \cap S'_d$, then this is the zero set of a section of νS_d so $e(\nu S_d) = PD([V])$.

Claim 7.4. *The class $[S_1] \in H_4(\mathbb{CP}^3; \mathbb{Z}) \cong \mathbb{Z}$ is a generator and $[S_d] = [S'_d] = d \cdot [S_1]$.*

Proof. For this claim we will make use of the Gysin sequence [MS74, p. 143] which for an oriented rank n vector bundle E over a manifold M is the exact sequence

$$\cdots \longrightarrow H^{i-n}(M; \mathbb{Z}) \xrightarrow{\cup e} H^i(M; \mathbb{Z}) \xrightarrow{\pi_0^*} H^i(E_0; \mathbb{Z}) \longrightarrow H^{i-n+1}(M; \mathbb{Z}) \longrightarrow \cdots$$

where $e = e(E)$ is the Euler class of the vector bundle, E_0 is the total space of the vector bundle without the zero section, and π_0 is the projection map of the vector bundle restricted to E_0. If we apply this to $\mathcal{O}(-1)$, the tautological line bundle over $\mathbb{CP}^n$ (see the proof of lemma 6.1), we first notice that E_0 is the set of all pairs (L, v) where L is a line through the origin in $\mathbb{C}^{n+1}$ and $v \in L$ is a non-zero vector on that line. Therefore we can identify E_0 with $\mathbb{C}^{n+1} \setminus \{0\}$, which we know is homotopy equivalent to S^{2n+1}, the $2n+1$ sphere.

This gives us $H^i(E_0; \mathbb{Z}) \cong \{0\}$ for all $0 < i < 2n+1$ and the Gysin sequence gives us exact sequences

$$0 \longrightarrow H^i(\mathbb{CP}^n; \mathbb{Z}) \xrightarrow{\cup e} H^{i+2}(\mathbb{CP}^n; \mathbb{Z}) \longrightarrow 0$$

for $0 \leq i \leq 2n-2$. Applying this to the case $n = 3$ we have

$$\mathbb{Z} \cong H^0(\mathbb{CP}^3; \mathbb{Z}) \cong H^2(\mathbb{CP}^3; \mathbb{Z}) \cong H^4(\mathbb{CP}^3; \mathbb{Z}) \cong H^6(\mathbb{CP}^3; \mathbb{Z})$$

since $\mathbb{CP}^3$ is connected. Furthermore we see that $e, e \cup e$ and $e \cup e \cup e$ generate the 2nd, 4th and 6th cohomology modules respectively.

Now for any complex vector bundle, we can endow it with a hermitian metric and we get an isomorphism $\overline{E} \cong \operatorname{Hom}_{\mathbb{C}}(E, \mathbb{C})$ by the map $v \longmapsto \langle\ , v\rangle$. Since $\mathcal{O}(1) = \operatorname{Hom}_{\mathbb{C}}(\mathcal{O}(-1), \mathbb{C})$, we have $e(\mathcal{O}(-1)) = c_1(\mathcal{O}(-1)) = -c_1(\mathcal{O}(1))$ and therefore $c_1(\mathcal{O}(1))$ is the other generator for $H^2(\mathbb{CP}^3; \mathbb{Z})$.

Since S_1 is a hyperplane in $\mathbb{CP}^3$, we know from proposition 6.2 that $PD([S_1]) = c_1(\mathcal{O}(1))$ and since PD is an isomorphism, $[S_1]$ is a generator for $H_4(\mathbb{CP}^3; \mathbb{Z})$.

Now consider the map

$$\begin{aligned} \phi : H_2(\mathbb{CP}^3; \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ [S] &\longmapsto \left|S \cap \mathbb{CP}^1\right|. \end{aligned}$$

where S is an embedded surface. If S_a is cut out by a degree a polynomial, then

$$\phi([S_a] + [S_b]) = \left|(S_a \cup S_b) \cap \mathbb{CP}^3\right| = a + b = \phi([S_a]) + \phi([S_b])$$

by Bezout's theorem and since for the generator $[S_1]$, $\phi([S_1]) = 1$, ϕ is an isomorphism. From this we see that $\phi([S_d]) = d = \phi([S'_d])$ and thus $[S_d] = [S'_d] = d \cdot [S_1]$, proving the claim. ☐

With this claim in hand we can now calculate $e(\nu S_d)$ which we know is equal to $c_1(\nu S_d)$. Since S_d and S_d' are compact submanifolds of $\mathbb{CP}^3$ that intersect transversally, we have $PD([S_d \cap S_d']) = i^*PD([S_d'])$ where $i : S_d \to \mathbb{CP}^3$ is the inclusion map (see [Bre93, p. 371]). From the definition of the Euler class above we have

$$c_1(\nu S_d) = PD([S_d \cap S_d']) = i^*PD([S_d']) = di^*PD([S_1]) = dx$$

where $x = i^*g = i^*PD([H])$ for any hyperplane $H \subset \mathbb{CP}^3$, proving the lemma. ∎

Lemma 7.5. *The Chern classes of S_d are given by*

$$c_1(S_d) = (4-d)x \quad \text{and} \quad c_2(S_d) = (d^2 - 4d + 6)x^2$$

where $x = i^*g$ and $g = c_1(\mathcal{O}(1))$ as in the previous lemma. Moreover, $\langle x^2, [S_d]\rangle = d$ and thus $c_2[S_d] = (d^2 - 4d + 6)d$ and $c_1^2[S_d] = (4-d)^2 d$. Additionally, Q_{S_d} is even if and only if d is even.

Proof. If we restrict the tangent bundle $T\mathbb{CP}^3$ to S_d, we can write it as the Whitney sum of the tangent bundle TS_d and the normal bundle νS_d, that is $T\mathbb{CP}^3\big|_{S_d} = TS_d \oplus \nu S_d$. Note that TS_d is a rank 2 complex vector bundle and νS_d is of rank 1. It is important here to clarify what we mean by 'restricting the tangent bundle.'

Consider the inclusion map $i : S_d \to \mathbb{CP}^3$. Then the restriction $T\mathbb{CP}^3\big|_{S_d}$ as a vector bundle over S_d is the pullback $i^*(T\mathbb{CP}^3)$. That is given a point $p \in S_d$, the fibre over p is the tangent space of $\mathbb{CP}^3$ at $i(p)$.

Note that $i : S_d \to \mathbb{CP}^3$ is covered by a bundle map $\varphi : T\mathbb{CP}^3\big|_{S_d} \to T\mathbb{CP}^3$, which takes a fibre T_p in $T\mathbb{CP}\big|_{S_d}$ to the fibre $T_{i(p)}$ in $T\mathbb{CP}^3$ via the identity map. Since these two vector bundles have the same rank, we know $i^*c(T\mathbb{CP}^3) = c(T\mathbb{CP}^3\big|_{S_d})$.

Using this and the Whitney product formula, we get

$$c(T\mathbb{CP}^3\big|_{S_d}) = i^*(1+g)^4 = (1+x)^4 = (1 + c_1(S_d) + c_2(S_d)) \cdot (1 + c_1(\nu S_d)).$$

and therefore

$$1 + c_1(S_d) + c_2(S_d) = (1+x)^4(1 + c_1(\nu S_d))^{-1} = (1 + 4x + 6x^2) \cdot (1 - c_1(\nu S_d) + c_1^2(\nu S_d)).$$

Note that $x^3 \in H^6(S_d; \mathbb{Z})$ and $x^4 \in H^8(S_d; \mathbb{Z})$ and since S_d is a real 4-manifold, these are zero. Lemma 7.3 tells us that $c_1(\nu S_d) = dx$ and we get

$$1 + c_1(S_d) + c_2(S_d) = (1 + 4x + 6x^2) \cdot (1 - dx + d^2x^2).$$

By comparing the dimensions of the terms we get $c_1(S_d) = (4-d)x$ and $c_2(S_d) = (d^2 - 4d + 6)x^2$ as desired.

To calculate $\langle x^2, [S_d]\rangle$ we have

$$\begin{aligned}
\langle x^2, [S_d]\rangle &= \langle (i^*g)^2, [S_d]\rangle \\
&= \langle g^2, i_*[S_d]\rangle \\
&= \langle g^2, PD(i_*[S_d]) \cap [\mathbb{CP}^3]\rangle \\
&= \langle g^2 \cup PD(i_*[S_d]), [\mathbb{CP}^3]\rangle \\
&= \langle g^2 \cup dPD(i_*[S_1]), [\mathbb{CP}^3]\rangle \\
&= \langle g^2 \cup dg, [\mathbb{CP}^3]\rangle \\
&= d\langle g^3, [\mathbb{CP}^3]\rangle \\
&= d
\end{aligned}$$

In the last step, we evaluated $\langle g^3, [\mathbb{CP}^3]\rangle = 1$, which needs justification. One way to see why this is the case is through intersection theory. We know that $g = PD(i_*[H])$, the Poincaré dual of the class of a hyperplane, so $g \cup g \cup g = PD(i_*[H_1 \cap H_2 \cap H_3])$ where H_1, H_2 and H_3 are three hyperplanes that intersect transversally. Since three transverse hyperplanes will intersect at a point in $\mathbb{CP}^3$, $g^3 = PD(i_*[\mathrm{pt}])$ and the Poincaré dual of a point is exactly the dual element to $[\mathbb{CP}^3]$, that is $\langle PD(i_*[\mathrm{pt}]), [\mathbb{CP}^3]\rangle = 1$. We can also see this using de Rham cohomology, where the Poincaré dual to a point is a bump top cohomology form, and the expression $\langle PD(i_*[\mathrm{pt}]), [\mathbb{CP}^3]\rangle$ corresponds to integrating this bump form over the whole manifold, giving 1 [BT82, p. 68].

From the calculation above we note that for odd d the term $\langle x^2, [S_d]\rangle = Q_{S_d}(x, x)$ is odd and hence Q_{S_d} is odd as well. If d is even, then $c_1(S_d) = (4-d)x \equiv 0 \mod 2$, implying $w_2(S_d) = 0$ and thus Q_{S_d} is even (corollary 6.9). Therefore Q_{S_d} is even if and only if d is even. For the Chern numbers we have

$$c_2[S_d] = \langle c_2(S_d), [S_d]\rangle = (d^2 - 4d + 6)\langle x^2, [S_d]\rangle = (d^2 - 4d + 6)d$$

and

$$c_1^2[S_d] = \langle c_1^2(S_d), [S_d]\rangle = (4-d)^2\langle x^2, [S_d]\rangle = (4-d)^2 d,$$

completing the proof of the lemma. ■

Theorem 7.6. *The hypersurface S_d is a smooth, simply connected, complex surface. The intersection form Q_{S_d} is equivalent to*

- $\lambda_d\langle 1\rangle \oplus \mu_d\langle -1\rangle$ *where* $\lambda_d = \frac{1}{3}(d^3 - 6d^2 + 11d - 3)$ *and* $\mu_d = \frac{1}{3}(d-1)(2d^2 - 4d + 3)$ *if d is odd, and*
- $l_d(-E_8) \oplus m_d H$, *where* $l_d = \frac{1}{24}d(d^2 - 4)$ *and* $m_d = \frac{1}{3}(d^3 - 6d^2 + 11d - 3)$ *if d is even.*

Proof. The implicit function theorem tells us that S_d is a smooth 4-manifold, and lemma 7.2 showed that it is simply connected.

By lemma 4.3 we know Q_{S_d} is unimodular so if Q_{S_d} is indefinite it suffices to compute the parity, rank and signature of the form, as per Lemma 4.11. We will see that only the $d = 1$ case will not be indefinite, and we deal with that case separately at the very end of the proof.

We already know Q_{S_d} is even if and only if d is even (lemma 7.5) and from lemma 6.6 we know the Chern numbers $c_1^2[S_d]$ and $c_2[S_d]$ satisfy

$$c_2[S_d] = \chi(S_d) = 2 + \mathrm{rk}(Q_{S_d}) \quad \text{and} \quad c_1^2[S_d] = 3\sigma(Q_{S_d}) + 2\chi(S_d).$$

We have $c_2[S_d] = (d^2 - 4d + 6)d$ and $c_1^2[S_d] = (4-d)^2 d$ from lemma 7.5 and using this we are now able to compute b_2^+, b_2^-, $\sigma(Q_{S_d})$ and $\mathrm{rk}(Q_{S_d})$ and we get

$$\mathrm{rk}(Q_{S_d}) = c_2[S_d] - 2 = d^3 - 4d^2 + 6d - 2$$

and

$$\sigma(Q_{S_d}) = \frac{1}{3}(c_1^2[S_d] - 2c_2[S_d]) = \frac{d}{3}(4 - d^2).$$

Since Q_{S_d} is unimodular, We know $b_2^+ + b_2^- = \mathrm{rk}(Q_{S_d})$ and $b_2^+ - b_2^- = \sigma(Q_{S_d})$. Solving these two equations simultaneously we get

$$b_2^+ = \frac{1}{3}(d^3 - 6d^2 + 11d - 3) \quad \text{and} \quad b_2^- = \frac{1}{3}(d-1)(2d^2 - 4d + 3).$$

Now, using lemma 4.11 and the fact that $|\sigma(Q_{S_d})| = -\sigma(Q_{S_d})$ for $d \geq 2$, we conclude that Q_{S_d} is equivalent to

$$b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle = \frac{1}{3}(d^3 - 6d^2 + 11d - 3)\langle 1\rangle \oplus \frac{1}{3}(d-1)(2d^2 - 4d + 3)\langle -1\rangle$$

if d is odd, and

$$\frac{\sigma(Q_{S_d})}{8}E_8 \oplus \frac{\mathrm{rk}(Q_{S_d}) - |\sigma(Q_{S_d})|}{2}H = \left|\frac{\sigma(Q_{S_d})}{8}\right|(-E_8) \oplus \frac{1}{2}(b_2^+ + b_2^- - (b_2^- - b_2^+))H$$
$$= \frac{1}{24}d(d^2-4)(-E_8) \oplus \frac{1}{3}(d^3 - 6d^2 + 11d - 3)H$$

if d is even. Note here that $d = 1$ is the only integer where b_2^+ or b_2^- are zero and thus Q_{S_1} is not indefinite so lemma 4.11 doesn't hold. However, we see that $\sigma(Q_{S_1}) = 1$ and since Q_{S_1} is unimodular, we have that $Q_{S_1} \cong \langle 1 \rangle$ and the theorem still holds. This completes the proof of the main theorem. ■

As a final remark it is definitely worth noting when X is a simply connected 4-manifold, $H_1(X;\mathbb{Z}) \cong H_3(X;\mathbb{Z}) \cong \{0\}$ and there is no torsion in any of the homology or cohomology modules as noted in corollary 3.6. In this case, simply knowing the rank of the intersection form completely determines all the homology and cohomology modules. For example consider S_4. Since $\mathrm{rk}(Q_{S_4}) = 22$, we know

$$H_0(S_4;\mathbb{Z}) \cong H_4(S_4;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(S_4;\mathbb{Z}) \cong H_3(S_4;\mathbb{Z}) \cong \{0\} \quad \text{and} \quad H_2(S_4;\mathbb{Z}) \cong \mathbb{Z}^{\oplus 22},$$

and Poincaré duality gives us the cohomology modules. To me, this is remarkable.

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