Math Circles - Surfaces

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Introduction

This instalment of math circles is intended to give you a taste of an area of pure mathematics that you (hopefully) haven't seen or even had a sniff of before. The layout of the course will be as follows. We will first create a universe with its laws and explore a few examples of objects that might exist in there. This universe, the universe of 2-dimensional surfaces, will seem extremely poorly lit at first and we will feel like we are walking around in the dark. So we will do the only thing we can do, walk around and see what happens.

Along our explorations we will bump into some surprising (sometimes terrifying) things and occasionally we will stumble upon a light switch, which will clear things up at least until that next dark corner. With a bit of luck, by the end of the three sessions our beautiful universe will be completely lit up in all its glory for us to see.

Because this area of mathematics is intended to be brand new to just about all of you, you do not need any background in just about anything to follow what is going on here. All you will need is your imagination and a willingness to explore.

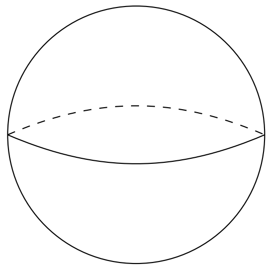
Before we begin it should be noted that this topic was first introduced to me by Ben Burton (who's currently at the University of Queensland in Australia) and the way I will be presenting it is almost entirely inspired by the way he taught it to me.

Let's go!

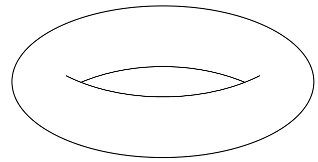
The Universe of 2-Dimensional Surfaces

Let's create the universe we are going to be exploring. Our universe that we will interested in is the universe of 2-dimensional surfaces. So, what kind of objects exist in our universe and how are we allowed to play with them? What are the "laws of physics" in this universe?

Two examples of such objects are the surface of a **sphere** and the surface of a donut (a **torus**), but we only care about the surface! You should picture these in your head as being hollow, so in some sense they are 2-dimensional things (since the surface is 2-dimensional, I will explain this a little more later on).



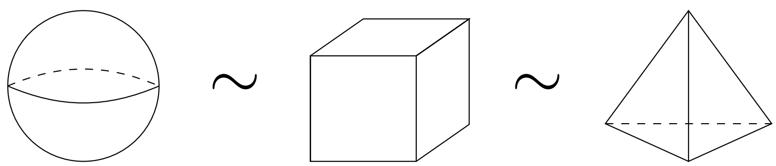
Sphere



Torus

So, what are we allowed to do with these objects? It turns out that the objects in our universe are made of spandex (or at least you can think of them that way). That is, we are allowed to stretch, shrink, squish, and deform our surfaces in any way we like. However, we **CANNOT** cut or glue our surfaces.

For example, the sphere (in this universe) can be legally deformed into a cube, or a tetrahedron since we can just squish the sphere until it looks like one of these without cutting or gluing. We can also stretch it out into a long snake, and then return it to a sphere.



Some legal deformations of the sphere

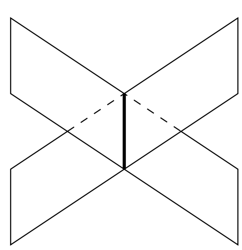
The torus can be legally deformed into a coffee cup (remember, we're only considering the surface of it) and back. The hole in the torus becomes the hole in the handle of the coffee cup.

So, now that we have some examples and maybe a bit of instinct as to what's allowed and what is not, let's define this universe a little more concretely.

Definition. A **2-dmensional surface** is an object that, when you get sufficiently close, just looks like a bent piece of a 2-dimensional plane.

This idea is something we're very familiar with on the surface of the Earth. If we get sufficiently close to the surface of the Earth (say about 180 centimetres or so) and look around, the surface of the Earth looks a lot like a bit of 2-dimensional plane (in fact so much so that the human race used to think it *was* a plane, and not a sphere!).

An example of something which is *not* a 2-dimensional surface is the object pictured below. If you zoom in to any point other than the intersection and look around, it sure looks like a piece of 2-dimensional plane. However, if we zoom in to any point on the intersection and look around, it always looks like 2 planes intersecting, it doesn't matter how close we get.



Not a 2-dimensional surface

Definition. Two surfaces, call them S and T, are **equivalent**, and we write $S \sim T$, if we can get one from the other.

So, we know the sphere is equivalent to the cube, and the torus is equivalent to a coffee cup! Now, before we get down to exploring this universe there are a couple of things worth noting.

- We are only concerned with surfaces that are not infinite and do not have boundary.
- We are not confined to the surfaces living in 3-dimensions.

What do these two points mean exactly? Well, not being infinite means we don't go off infinitely far in any direction. Not having a boundary means that I can walk along the surface and never reach an edge (which I cannot do on a piece of paper for example, which means a piece of paper has a boundary). Not being confined to 3-dimensions means that we are allowed to have two 'bits' of a surface pass through each other. Even though this might seem like we are cutting our surface to allow it to pass through itself, this is just a 3-dimensional representation of what is going on. If we allow more dimensions, then a surface can be legally deformed through a step that appears to be passing through itself, without passing through itself at all (there's enough room to go around instead of passing through). Weird huh!?

Now that we have set up and explored our universe a little (in possibly a more confusing than enlightening manner), let's get down to business!

Big Questions

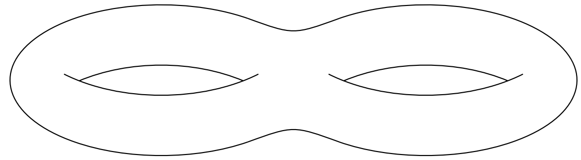
Here are some questions that we will be setting out to answer. These are the big questions we should be keeping in the back of our minds throughout these three weeks.

1. If I give you any two surfaces, can you tell me whether or not they're equivalent? For example, is the torus equivalent to the sphere? We hope they aren't, but how can we prove it?
2. If I give you a surface, can you tell me which surface it is?
3. Can we make a list of all possible surfaces?

With these questions in mind, let's start looking around in our universe!

Some more 2-dimensional surfaces

We have already seen a sphere (which we will call S) and a torus (T). But what other surfaces are there? Well, we can have a 2-holed torus, or a 3-holed torus, or maybe even an n-holed torus! But is that all, and are they even different? Let's keep looking.



A 2-holed torus

Projective Plane

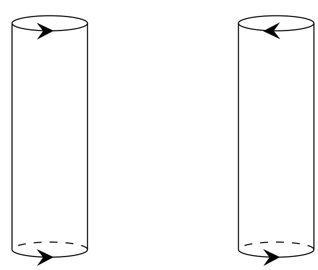
This surface is an interesting one, because we cannot visualise it living in 3-dimensional space (but it does exist in 4-dimensions in our universe!). We will denote it by PP, and here's how we make it.

First, we take a Möbius strip and a disk. We know a Möbius strip has only 1 edge (if you don't believe me, make your own Möbius strip and draw a line along one of the edges and see what happens), and the disk has one edge. Well, let's glue them together! It turns out that this is a 2-dimensional surface and it does live in our universe.

Klein Bottle

Again, like the projective plane, this cannot be visualised in 3-dimensional space without it passing through itself, but it does exist in 4-dimensions! To describe it, we must first take a look at the torus in a little more detail.

One way to think about making a torus is to glue two ends of a cylinder together, but what if you glued the two ends together in opposite directions (as shown in the diagram below, where the arrows mean glue the edges together so that the arrows match up)? Then you would get a Klein bottle, which will be denoted KB.



Torus *Klein Bottle*

New Surfaces from Old

So now we have four examples of surfaces that live in our universe. We still have no idea (although we can guess) whether or not these are equivalent or different, but at least we have something! Now we will explore two apparently different ways of creating new surfaces from old ones.

Adding a Handle

This process is a lot like what it sounds like. Given a surface (call it Bob), we are going to take it from our universe, attach a handle and put it back! Here's how we attach a handle:

1. Cut two holes in Bob. This gives us two edges which are ready to be glued to something.

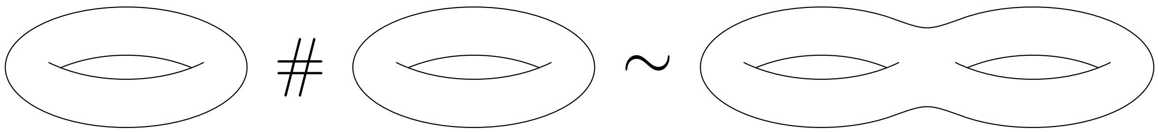
2. Take a cylinder and glue each of the two edges to the two edges on the surface.

Voila! You now have a potentially new surface. See if you can convince yourself that adding a handle to a sphere gives you a torus.

Connected Sum

This is a kind of way of joining two surfaces together. Let's say now we have two surfaces, Jack and Jill, and we want to create a new surface out of these two. Here's what we can do. We can cut a hole in both Jack and Jill, which gives both surfaces an edge, and then glue them both together along that edge! We will call this surface Jack#Jill.

For example, if we take two tori and take their connected sum, with a bit of imagination (try it!) we can see that



$T\#T \sim$ 2-holed torus.

Spotting the Difference

We have a bunch of surfaces now, and we would really like to know whether or not they are equivalent. We guess that the sphere and the torus are not equivalent, but right now we have no way of proving that. If we are to prove it we need some thing, some quantity, number, colour, face, anything that stays the same under legal deformations of the surface. Then if we calculate such a quantity to be, say 7 for the torus and 5 for the sphere, then we know that we can never deform a torus to be a sphere, and we will have proved that they are indeed not equivalent. Such a tool is called an **invariant**. Although this gives us a tool to tell if two things are different, it **does not** tell us when two things are the same. Let's create an invariant now.

Euler Characteristic

We will now learn how to calculate the Euler characteristic of a surface. This will be a number that we can associate to every 2-dimensional surface that is an invariant, and here's how we calculate it.

1. Deform your surface so that it is some sort of polyhedra. That is, deform it until the surface is made up of polygons. For example, we can legally deform the sphere to be a cube, or a tetrahedron.

2. Count the number of vertices (V), edges (E), and faces (F) on your polyhedra.

3. Calculate the quantity $V - E + F$, and this is your **Euler characteristic**. If your surface is A, then to denote the Euler characteristic we write $\chi(A)$.

Let's do an example with the sphere. As we've seen before, the sphere is equivalent to a cube and a cube has 8 vertices, 12 edges and 6 faces. Therefore we have

$$\chi(S) = V - E + F = 8 - 12 + 6 = 2.$$

At this point you should be screaming in protest! This only works if we turn a sphere into a cube, there's no way we will get the same number if we turn the sphere into anything else, right? Amazingly, this is not the case. Let's have a look at what happens if we turn the sphere into a tetrahedron instead. We know a tetrahedron has $V = 4$, $E = 6$, $F = 4$, so we see that $\chi(S) = 4 - 6 + 4 = 2$, which is the same number we got with the cube! Amazing.

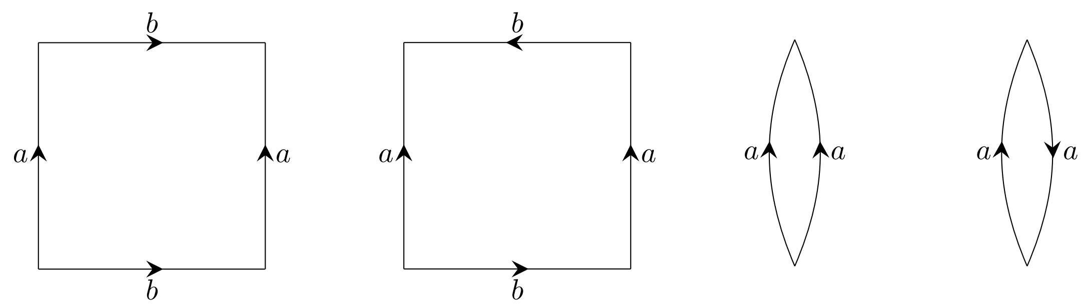
Fact 1. *The Euler characteristic is an invariant of any surface.*

With all of this in mind, we now have the tools to prove that two surfaces are not equivalent. How can we prove a surface is not equivalent to a sphere? Easy, we calculate its Euler characteristic and if it is not 2, then we know the surface is not equivalent to the sphere. Unfortunately, if we find the Euler characteristic of our surface is 2, then we cannot say anything about whether or not it is equivalent to the sphere.

Planar Models

So, now that we have the Euler characteristic, in theory we should be able to start proving some things about surfaces. However, in practice this is not turning out to be the case. The current way of calculating the Euler characteristic of a surface seems prohibitively tedious and difficult, so we need some help. Enter planar models!

This is a way of visualising surfaces without having to imagine 4 dimensions. We will see it is a useful tool for just about everything! Let's have a look at the planar models of some familiar surfaces (see if you can work these out for yourself, some of them appear in the exercises).



Torus *Klein Bottle* *Sphere* *Projective Plane*

Word Representations

Drawing these planar models out every time does help for cutting and pasting, but it does sometimes get tedious. Instead we can simply encapsulate all the information of a planar model from a word representation. So, how do you make a word representation you ask? As follows!

1. Draw your surface as a single polygon.
2. Start at any vertex, move in any direction (clockwise orcounter-clockwise).
3. Read off the letters and every time you have an arrow going in the opposite direction you are, put an inverse sign.

So, we have some word representations as follows:

$$\begin{aligned} \text{Torus: } & aba^{-1}b^{-1} \\ \text{Klein bottle: } & abab^{-1} \\ \text{Projective Plane: } & aa \\ \text{Sphere: } & aa^{-1} \end{aligned}$$

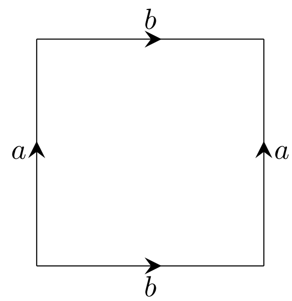
At this point you might be wondering, why do we even care about all of this. Sure, it makes it easier to visualise, but it doesn't seem to be getting us any closer to proving anything. Or maybe it does! Here are two reasons why we care about planar models and their word representations.

1. It makes calculating the Euler characterististic easier, and

2. For the first time, we actually have a chance of proving that two surfaces are the same!

Computing the Euler characteristic

Let's look at the first advantage with the Torus. From the exercises, we know in order to turn the Torus into a polyhedron, and then count the numbers of vertices, edges and faces without making a mistake, we would probably have to be some sort of Greek god. However, with a planar model it becomes much easier. Let's look at it again.



At first glance, if we count the number of vertices on this picture we get 4. However, this is incorrect because when we glue all the arrows back together, we find that some of these vertices are actually glued together and thus are the same. Let's count these slowly.

The vertex at the top left is at the 'end' of the a edge, thus it is glued to the top right vertex, so those two are the same. The top right vertex is also the 'end' of the b edge, so it is the same as the bottom right vertex. The bottom right vertex is also the''start' of the a edge, so it is the same as the bottom left vertex. Thus, all the vertices are the same and there is only 1 vertex.

If we count the edges, we see that the two a edges are actually the same (after we glue it back up) and the two bs are also the same. Thus there are 2 edges.

Of course, there is only one face, the square itself. So, with this in mind we have

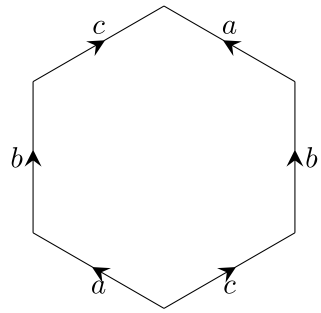
$$\chi(T) = V - E + F = 1 - 2 + 1 = 0.$$

Much easier!

Proving the equivalence of two surfaces

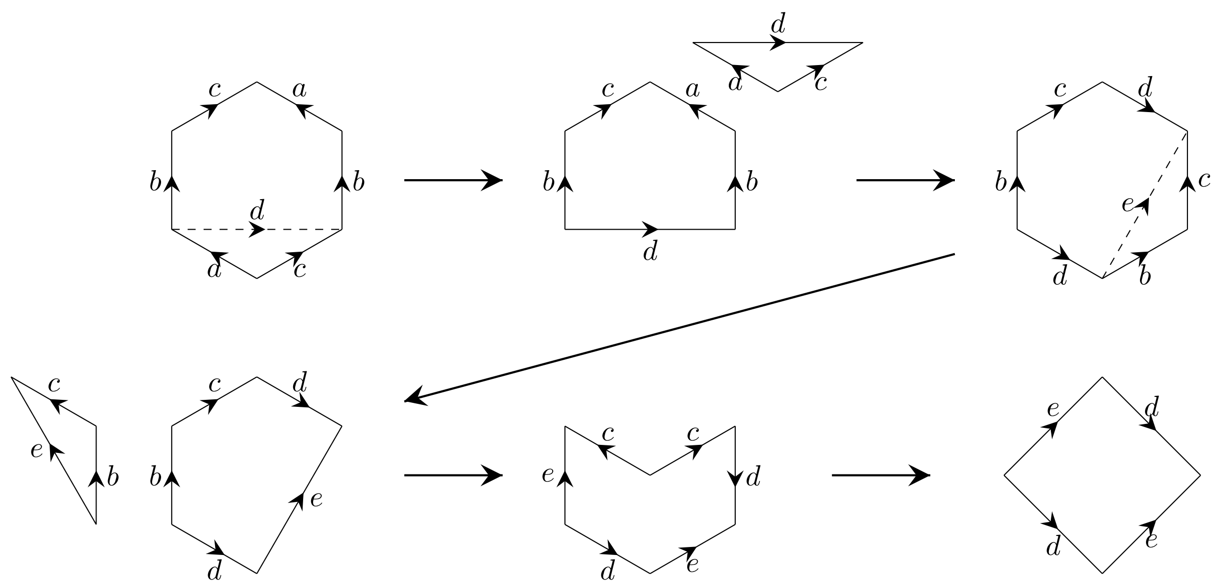
So far, we've only had ways of proving that two surfaces are *not* equivalent. Let's now do an example to see how we can use a technique called cutting and pasting (very high-tech stuff!) to actually show two surfaces which appear to not be equivalent are actually equal.

Here's a question, what is the surface with word representation $abca^{-1}b^{-1}c^{-1}$? Is it something we've seen before or can we prove it's something new? Let's start by looking at it's planar model to figure out it's Euler characteristic. If it doesn't agree with the Euler characteristic of anything we've already seen, we can definitely say it's not a surface we've seen before. It's planar model is:



This has 2 vertices (top right, top left and bottom are all the same, as are the other three), 3 edges and 1 face, so its Euler characteristic is 0. Unfortunately, we can't conclude anything from this, but we can make a guess that maybe it's the Klein bottle or the torus (it can't be anything else we've seen because nothing else has Euler characteristic 0).

Well, let's start playing around to see if it is. The general strategy is as follows: make cuts between vertices, keeping track of which edges are to be glued back together, and glue other vertices back together. This isn't a very clear description, but it will become clear with this example. We will perform the following cuts and pastes on our planar model, and see what happens! The dotted lines are where we are going to cut.



Now this last shape looks extremely familiar! It's a torus and what we have shown here is that

$$abca^{-1}b^{-1}c^{-1} \sim ede^{-1}d^{-1} \sim \text{ a torus!}$$

Amazing! It's a fun exercise to try and imagine how gluing the edges from the first hexagon actually gives you a torus (we've shown it indeed does!).

Orientability

So we've played around with cutting and pasting a little, but we're still not satisfied. This is because we know

$$\chi(T) = \chi(KB) = 0.$$

This doesn't tell us that these are different, but if we take a planar model for the Torus and try and cut and paste it into a planar model for a Klein bottle, we really struggle to do so. Remember, just because we haven't been able to doesn't mean it's impossible, but our guts tell us it is. We need something else besides the Euler characteristic. It's extremely frustrating that we can't seem to tell the torus and Klein bottle apart.

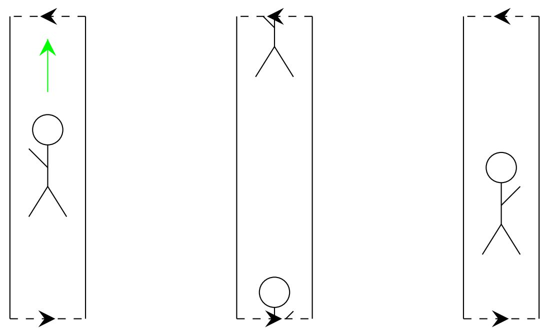
Enter, orientability.

Notice that when we tried cutting and pasting the planar model of the torus, at every step if we look at the word representation, we always have letters matching up with opposite orientations. For example, the two word representations we have seen for the torus are

$$aba^{-1}b^{-1} \quad \text{and} \quad abca^{-1}b^{-1}c^{-1}$$

and in both cases, we have exactly one of each pairs of letters going forward, and the other backward. However, this isn't the case with the standard word representation of the Klein bottle, $abab^{-1}$. So, what's going on here? It turns out the answer lies in looking at the Möbius strip more carefully.

Picture a little 2-dimensional man with one arm (his right one) living in a Möbius strip (here he actually is in the Möbius strip, not above or below it, but in it, much like we are inside our 3-dimensional universe). If he walks (as best as a 2-dimensional man can manage) around the Möbius strip once, when he gets back to the same spot, his right hand will be missing and he will now have a left one! Somehow the notion of right handed and left handed does not make sense when we're dealing with a Möbius strip, or when we're dealing with any surface with a copy of the Möbius strip inside it.

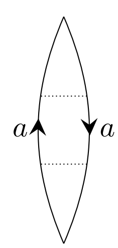


Being left or right handed doesn't make sense on a Möbius strip

Let's define this formally.

Definition. A surface is **non-orientable** if it contains a copy of the Möbius strip. It is **orientable** otherwise.

So, is the projective plane orientable? Well, we built it out of a Möbius strip so we know there's a copy of it living somewhere in the surface. So the answer is "no". However, more enlightening is looking at the planar model. We see that between the dotted lines below, lives a Möbius strip!



A Möbius strip in a projective plane

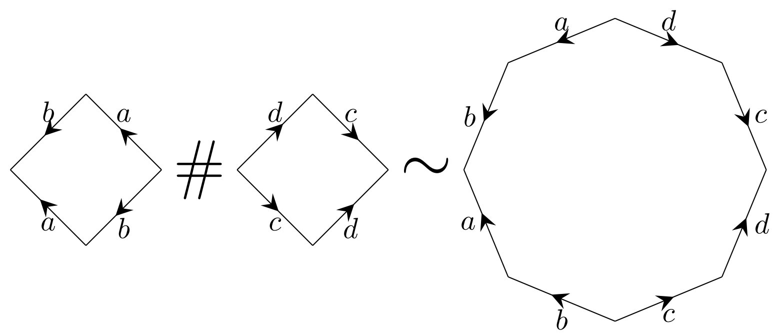
In fact, if you think hard enough and do the exercises, we can see that if a word representation of a surface has a pair of letters, both going forwards or backwards, then the surface is not orientable! For example, the surface $aabcdd^{-1}cb^{-1}$, whatever it is, is not orientable because it has a pair of cs with the same direction (it also has the pair of as but we only need one pair for it to be non-orientable).

Connected sum and the Euler characteristic

One of the important exercises was asking about planar models for the connected sum, and how the Euler characteristic changes when we take the connected sum of two surfaces.

Well, after a bit of work in the exercises you can convince yourself that if we have two word representations W and X for two surfaces A and B, then a word representation for $A\#B$ is WX. You simply squish the two words together! This can be thought of in terms of the planar models as follows: Take the vertex from A that you start the word from, the vertex in B that you start the word from, glue the two models together at the point, and stretch the surface to separate the point and make a single polygon.

For example, if we take two tori, say given by $aba^{-1}b^{-1}$ and $cdc^{-1}d^{-1}$, then a word representation for $T\#T$ (which we're pretty sure is the torus with two holes) is $aba^{-1}b^{-1}cdc^{-1}d^{-1}$! To see this with planar models we have:



So, with this in mind, we can now generalise this idea and (by one of the exercises) show the following theorem.

Theorem 2. *If A and B are any two surfaces, then $\chi(A\#B) = \chi(A) + \chi(B) - 2$.*

Pretty neat hey? It finally feels like we're starting to make some progress.

What we have so far

Phew! Okay, let's stop and take a look at what we know so far. Using our formula for the connected sum and the Euler characteristic, as well as our newfound ability to spot whether or not something is orientable just by looking at a word representation or a planar model, we can make the following table. It's a great exercise to verify all the information below.

Surface	Euler Characteristic	Orientable?
Sphere	2	Y
Torus	0	Y
2-holed torus ($T\#T$)	-2	Y
n-holed torus	2 - 2n	Y
Klein Bottle	0	N
Projective Plane	1	N
$PP\#PP$	0	N
$PP\#PP\#PP$	-1	N
$PP\#KB$	-1	N
$PP\#T$	-1	N

Looking at this table we can now conclude a lot of things! The torus is indeed non equivalent to the Klein bottle (phew!), and the n-holed torus is not equivalent to the k-holed torus for any $k \neq n$. However, we still can't tell apart $PP\#PP$ and KB or $PP\#PP\#PP$, $PP\#T$ and $PP\#KB$. Maybe they're the same, or maybe we need more invariants. We will soon find out.

Cutting and Pasting with Words

Earlier on, we showed that the surface corresponding to the word $abca^{-1}b^{-1}c^{-1}$ is similar to the torus through a series of cutting and pasting polygons. Although this method in theory is useful, in practice it is cumbersome. We'd like to encode some of this in the form of legal word manipulations. Before we try to list what exactly we can do with words that keeps the corresponding surface the same, we should note a few more things. It doesn't matter where on the planar model we start our word, or whether we go clockwise or counter-clockwise. These kinds of things should be accounted for in a useful list of legal word manipulations. Let's try and compile such a *list of legal word manipulations.*

1. **Relabel edges:** For any particular edge, you can name it whatever you want. You can relabel an edge named a, "Bernice" if you want, just as long as you relabel both occurrences of the original a.

2. **Change direction:** For any particular edge, you can switch the direction of both occurrences of the same letter. For example, $aabbcc \sim aabbc^{-1}c^{-1}$.

3. **Cycle word:** As we noted, it doesn't matter where on a planar model you start reading off the labels of the edges to form your word. So, we're allowed to cycle words and, for example, we have that $abca^{-1}c^{-1}b^{-1} \sim c^{-1}b^{-1}abca^{-1}$.

4. **Read in the opposite direction:** This corresponds to reading around a planar model counter-clockwise instead of clockwise (say).

5. **Merge edges:** If the same sequence of letters appears as a group, then you can merge them into one. For example, $\dots bc \dots bc \dots \sim \dots d \dots d \dots$.

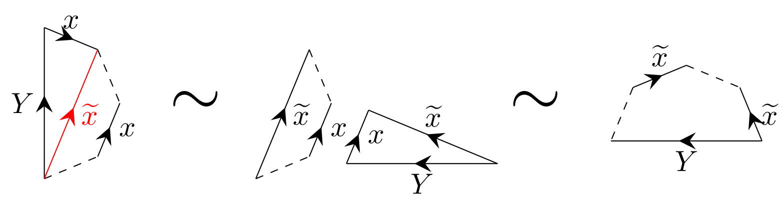
6. **Cancel aa^{-1} pairs (unless that's all you have):** This is what you'd expect and is what we did in the last step of showing $abca^{-1}b^{-1}c^{-1} \sim T$ above.

7. $\ldots Yxx \ldots \sim \ldots xY^{-1}x \ldots$.

8. $\ldots Yx \ldots x^{-1} \ldots \sim \ldots x \ldots x^{-1}Y \ldots$.

In 7 and 8, the capital Y corresponds to any sequence of letters. The legal manipulations 1 to 6 are all manipulations that are clear are legal. However, 7 and 8 do require some non-trivial cutting and pasting to guarantee that we don't start with a word, apply (say) 7 and end up with a word corresponding to a surface not equivalent to the original surface. In fact, making sure our words don't change enough to change the surface is the whole point of these rules. Let's try proving 8.

Proof. Let's say we have a planar model which is a polygon that has a word representative looking like $\ldots Yx \ldots x^{-1} \ldots$ where Y represents a whole bunch of edges. Then we can cut and paste that polygon as follows where the dashed lines represent other edges in the polygon which we're not particularly interested in, and we are cutting along the red edge.



The proof of rule 8

We can now see that the resulting polygon has a word representation of the form $\ldots x \ldots x^{-1}Y \ldots$, which completes the proof. ■

Rule 7 is proved in pretty much the same way, you just have to make a clever cut and paste. Try it for yourself!

The Manipulations in Action

So we've now created these word manipulations, which we've checked correspond to cutting and pasting planar models (which really was far too cumbersome for my liking). What these word manipulation rules have allowed us to do, is to perform the usual cutting and pasting and encode it in words! Hopefully we will now find it much easier to check whether two surfaces are the same now. Let's put it into action and see!

Let's try and attack the question of whether or not $KB \sim PP\#PP$. We have

$$KB \sim aba^{-1}b \sim aabb \sim PP\#PP$$

where the manipulation used rule 7, and we see that $KB \sim PP\#PP$! Amazing hey? See if you can perform the cut and paste (we only did it once, so there's only one) that turns the Klein bottle into the connected sum of two projective planes. This is promising, let's see what else we can work out!

Maybe we can settle this question about whether or not $PP\#KB \sim PP\#T \sim PP\#PP\#PP$.

$$\begin{aligned} PP\#T \sim aabcb^{-1}c^{-1} &\overset{7}{\sim} ab^{-1}acb^{-1}c^{-1} \\ &\overset{7}{\sim} ab^{-1}b^{-1}c^{-1}a^{-1}c^{-1} \\ &\overset{7,3}{\sim} aab^{-1}b^{-1}c^{-1}c^{-1} \\ &\overset{1}{\sim} aabbcc \sim PP\#PP\#PP. \end{aligned}$$

Whoa, now there's something unexpected! Let's stop and think about this for a sec. Knowing that $T\#PP \sim PP\#PP\#PP$ should freak you out a little (a lot really). What it tells us is that if we take a nice orientable surface like a torus, and then attach a projective plane, the non-orientability of the projective plane spreads through the torus, in some sense turning it into two projective planes! Weird. Another thing this tells us is that connected sum is not cancellative, that is, just because $A\#C \sim A\#B$ it **does not** mean $B \sim C$ (since we know $PP\#PP \not\sim T$).

Now we know from above that $KB \sim PP\#PP$, so we would expect that $KB\#PP \sim PP\#PP\#PP$, but let's check anyway.

$$PP\#KB \sim aabcbc^{-1} \overset{7}{\sim} aabbc^{-1}c^{-1} \sim PP\#PP\#PP.$$

Through these manipulations, we now know $PP\#PP \sim KB$ and, more surprisingly, $PP\#PP\#PP \sim KB\#PP \sim T\#PP$. Note here that both $PP\#PP$ and KB have the same Euler characteristic and orientability, as do $PP\#PP\#PP$, $T\#PP$, and $KB\#PP$. Coincidence? Maybe.

Big Questions - One More Time

So, we've been stumbling around in the dark for a while now, and it feels like we've found a few light switches. Our shins are getting sore from banging into all the furniture in the dark, so let's regroup and revisit the big questions we had at the beginning of our journey.

1. If I give you any two surfaces, can you tell me whether or not they are equivalent?
2. If I give you a surface, can you tell me which one it is?
3. Can we make a list of all possible surfaces?

Looking back at all the work we've done, cutting and pasting, developing the Euler characteristic and orientability, it seems like if the answer to these questions is still "no", we've at least made some progress. Let's try and investigate it further. Suppose we are given a random word corresponding to some surface. Absolutely any word at all, say $abcd^{-1}b^{-1}zz^{-1}wxwx^{-1}$.... Well, let's try and find some sort of standard form for it. We can do this in the following steps:

1. Collect like terms.

Say we have a word $WaXaY$ where W, X and Y are arbitrary strings of letters corresponding to multiple edges. Then by applying rule 7 repeatedly we have

$$WaXaY \sim WaaX^{-1}Y \sim aW^{-1}aX^{-1}Y \sim aaWX^{-1}Y$$

which tells us that if we have like terms, we have that it is equivalent to $PP\#\{\text{something}\}$.

2. Repeat 1 until you can't.
We can look for more like terms in the {something} and repeat the process above. Once we have run out of like terms, our word looks like

$$a_1 a_1 a_2 a_2 \cdots a_n a_n B$$

where B is some word without like terms, so it corresponds to an orientable surface. So we have that our surface is equivalent to $\overbrace{PP\#PP\#\cdots\#PP}^{n\text{-times}}\#$\{something orientable\}. If our word didn't have any like terms to begin with, we just skip the first two steps and go straight to step 3.

3. Group linkings of pairs.
So now we have gotten rid of all the edges which have the same sign appearing in both occurrences of their letter, we can start looking for pairs of edges which are interlocking. We are after things of the form $VaWbXa^{-1}Yb^{-1}Z$, where once again V, W, X, Y, Z are arbitrary strings of letters, which may or may not correspond to multiple edges. Then, using 8 repeatedly and sometimes cycling the word we get

$$\begin{aligned}
VaWbXa^{-1}Yb^{-1}Z &\sim aWbXa^{-1}VYb^{-1}Z \\
&\sim abXa^{-1}VYb^{-1}WZ \\
&\sim aba^{-1}VYXb^{-1}WZ \\
&\sim aba^{-1}b^{-1}WZVYX
\end{aligned}$$

which tells us that our original surface is

$$\overbrace{PP\#PP\#\cdots\#PP}^{n\text{-times}}\#T\#\{\text{some other orientable word}\}.$$

4. Repeat 3 until you can't.
Enough said. If we keep performing the steps in 3, we end up with a word representing a surface which is equivalent to

$$PP\#\cdots\#PP\#T\#\cdots\#T\#\{\text{some orientable word without pairs of interlocked edges}\}.$$

With a bit of thought you can convince yourself that if we have an orientable word where there are no interlocking edges, then it must be of the form $xx^{-1}yy^{-1}zz^{-1}$... for all the left over edges.

5. Cancel all remaining pairs.
So now we have a word which represents $PP\#\cdots\#PP\#T\#\cdots\#T$, which is pretty amazing. This tells us that whatever surface we have, if it is not a sphere (in which case we would be skipping right to step 5) then it is a connected sum of projective planes and tori! This is flabbergasting, but we can do even better.

6. Let the projective planes work their magic.
We know from earlier that $PP\#T \sim PP\#PP\#PP$, so every time we see $PP\#T$, we can replace it with $PP\#PP\#PP$.

So, if a surface is orientable, then we skip steps 1 and 2, and stop at step 5 (since step 6 doesn't have a chance to go into action) so we end up with a connected sum of tori or a sphere (the latter occurs if there are no linkings of pairs to begin with in step 3). If a surface isn't orientable, then the existence of one projective plane will spread like a virus and the surface will be equivalent to a connected sum of projective planes. Putting all of this together we get the following absolutely remarkable theorem, which was proved sometime around 1920.

Theorem 3 (Classification Theorem of Closed 2-Manifolds)**.** *Any 2-dimensional surface is equivalent to exactly one of the following three surfaces.*

- *A Sphere.*
- *A connected sum of projective planes (if it is non-orientable).*
- *A connected sum of tori (if it is orientable).*

With a bit of careful thought we can now convince ourselves that knowing the Euler characteristic and orientability of a surface is enough to tell us exactly what it is equivalent to!

Victory Lap

From stumbling around in the dark at the beginning, we have come a long way and now our entire universe is illuminated, or at least the parts we were interested in. With the classification theorem in our back pockets, we have the ultimate flashlight to enlighten us about any surface we are given.

This theorem to me is absolutely remarkable, and one of the most beautiful in all of mathematics. The fact that there is such a clean and neat classification of every possible surface blows my mind every time I think about it. It is results like this that makes math worth doing for its own sake.

Although we have been shown a clear view of what's going on, it is by no means the end of the road for studying surfaces. We can always ask more questions (as with anything in mathematics), like what 3-dimensional surfaces can exist? What about 4-dimensional, or even higher? How would we even define such a thing, and can we have some sort of "planar model" of these to help us?

If you are interested in learning more about this area of mathematics (topology and geometry), then a good place to start is the book *The Shape of Space* by Jeffrey Weeks. Good luck!