# RESEARCH STATEMENT

## ORDERED GROUPS AND MAPPING CLASS GROUPS (OF ALL SIZES)

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My research is in the area of geometric group theory and low dimensional topology, and I think about algebraic properties of symmetries of low-dimensional manifolds. In particular, I am interested in the mapping class group of both infinite-type and finite-type surfaces, and left-orderable and circularly-orderable groups. What follows is an account of selected research contributions of mine, along with future projects and questions I am planning to tackle. Section 1 involves mapping class groups (symmetries of 2-dimensional manifolds) and Section 2 is about left- and circularly-orderable groups (symmetries of 1-dimensional manifolds). The interplay between the two sections will be in the concluding Section 3.

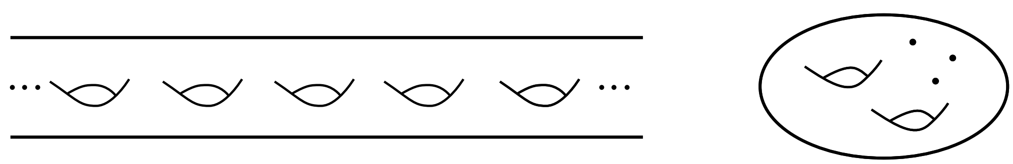

Figure 1. An infinite-type surface (the ladder) and a finite-type surface (a thrice punctured genus 2 surface)

## 1. All mapping class groups big and small

Intuitively, an infinite-type surface has infinitely many topological features, whereas a finite-type surface has only finitely many features (see Figure 1). Formally, a finite-type surface has finitely generated fundamental group. The *mapping class group* $\mathrm{MCG}(\Sigma)$ of a surface $\Sigma$ is a group of symmetries of the surface (specifically, the group of homeomorphisms up to isotopy). A mapping class group of an infinite-type surface is called a *big mapping class group*.

Mapping class groups (of the small variety) are ubiquitous in mathematics. For a few examples among many, they appear in 4-manifold topology through Lefschetz fibrations, and 3-manifold topology through Heegaard splittings and mapping tori. Mapping class groups are closely related to automorphism groups of free groups and right-angled Artin groups, and are a generalization of the braid group. The mapping class group is also the fundamental group of the moduli space of a Riemann surface, providing a bridge to algebraic and complex geometry.

The study of big mapping class groups has exploded in recent years. They naturally arise while studying complex dynamics and foliations of 3-manifolds, and some big mapping class groups are extensions of Thompson's groups. These groups are large and seemingly unmanageable (for example, they are not locally compact or compactly generated), yet there are many tools, both novel and adapted from the finite-type setting, which make studying these groups tractable. Miraculously, there is a classification of infinite-type surfaces, which has been repeatedly and successfully exploited to understand big mapping class groups.

1.1. **Mapping class groups and covering spaces.** In a series of papers in the early 1970s, Birman and Hilden studied the connection between mapping class groups and covering spaces, with a fruitful yeild! Given a covering of a surface $\Sigma$ by a surface $\widetilde{\Sigma}$ with mild restrictions, they proved that there is a strong relationship between a subgroup of the mapping class group of $\widetilde{\Sigma}$, called the *symmetric mapping class group* $\mathrm{SMCG}(\widetilde{\Sigma})$, and a subgroup of the mapping class group of $\Sigma$, called the *liftable mapping class group* $\mathrm{LMCG}(\Sigma)$. Intuitively, the symmetric mapping class group consists of mapping classes of $\widetilde{\Sigma}$ that project to $\Sigma$, and the liftable mapping class group consists of mapping classes of $\Sigma$ that lift to $\widetilde{\Sigma}$. More formally, Birman and Hilden proved the existence of a surjective homomorphism $\mathrm{SMCG}(\widetilde{\Sigma}) \to \mathrm{LMCG}(\Sigma)$ with kernel isomorphic to the deck group of the cover. The mild conditions on the cover were later removed due to results of Machlachlan-Harvey [25] and Kerckhoff [22].

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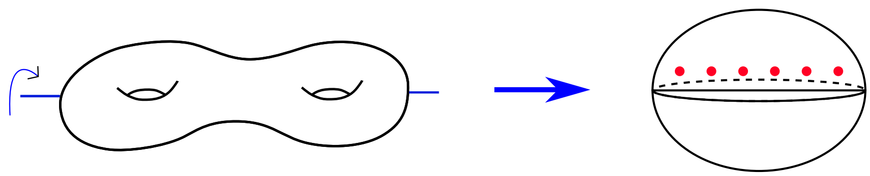

FIGURE 2. The quotient by a rotation by $\pi$ gives the hyperelliptic cover

In the case when the covering is the hyperelliptic cover of a genus 2 surface over the sphere (see Figure 2), Birman and Hilden showed that the symmetric and liftable mapping class groups concide with the full mapping class groups. The relationship between these two groups was then exploited to give the first group presentation for the genus 2 mapping class group [8]. This raises the natural question: for which covers is it true that every mapping class of the base surface lifts, or that every mapping class of the covering surface projects (or both)?

In their seminal Annals paper, Birman and Hilden state that whenever the cover is a cyclic branched cover of the sphere, every mapping class of the sphere lifts to the covering space [9, Theorem 5]. This Theorem is not correct (see the erratum [10]). I found a counterexample, and together with Rebecca Winarski, we provided a full classification of cyclic branched covers of the sphere with the property that every mapping class lifts. Our classification result relies on parametrizing cyclic branched covers over the sphere by tuples of elements of $\mathbb{Z}/n\mathbb{Z}$.

**Theorem 1** (Ghaswala-Winarski [20])**.** *An $n$-sheeted cyclic branched cover of the sphere at $k$ points has the property that every mapping class lifts if and only if the cover corresponds to the tuple $(1, \cdots, 1)$, $k \equiv 0 \mod n$, or the tuple $(1, -1)$.*

The techniques used in proving Theorem 1 have since been used to generalize our result to slightly broader families of branched covers over the sphere [1, 2, 3]. However, a full classification, even of abelian covers, remains elusive at the moment. The following question is an avenue of research for me.

**Question 2.** *Which finite-sheeted abelian branched covers of the sphere have the property that every mapping class lifts?*

In the case where the surfaces have non-empty boundary, Alan McLeay and I classified all finite-sheeted regular covers with the property that all mapping classes lift, or all mapping classes project. An important family of covers here are the *Burau covers*, which are a specific collection of cyclic branched covers of the disk.

**Theorem 3** (Ghaswala-McLeay [19])**.** *Suppose we are given a finite-sheeted regular cover of surfaces with non-empty boundary.*

- *All mapping classes of the base lift if and only if the cover is a Burau cover. The liftable mapping class group is a finite-index subgroup otherwise.*
- *All mapping classes of the cover project if and only if $\widetilde{\Sigma}$ is a disk, a cylinder, or $\widetilde{\Sigma}$ is a genus 1 surface with 1 boundary component and $p$ is a hyperelliptic cover. The index of the symmetric mapping class group is infinite otherwise.*

One of the tools used to prove Theorem 3 is the action of the mapping class group on a particular fundamental groupoid of the surface. This is a new technique in the study of mapping class groups with boundary.

1.2. **Braid groups and the Burau representation.** The *braid group* $B_n$ on $n$ strands, is isomorphic to the mapping class group of a disk punctured $n$ times. There is a standard embedding of the braid group $B_n$ into the mapping class group of a surface with genus at least $\frac{1}{2}(n-2)$. This embedding maps each standard braid generator to a special mapping class called a *Dehn twist*.

A question of Wajnryb [29] asks whether there are *non-geometric embeddings* of the braid group in a mapping class group, that is, an embedding of the braid group where each standard generator is not mapped to a Dehn twist. Non-geometric embeddings have since been constructed in the works of Bödigheimer-Tillman [12], Kim-Song [23], Song [26], Song-Tillman [27], and Szepietowski [28]. Alan McLeay and I contribute to the list of non-geometric embeddings by utilizing the Burau covers from Theorem 3.

**Theorem 4** (Ghaswala-McLeay [19])**.** *Let $n \geq 2$. For each $k \geq 2$, there exists a surface $\Sigma$ and an injective homomorphism $B_n \to \mathrm{MCG}(\Sigma)$ such that the image of a standard braid generator is a $(k-1)$-chain twist for all $i$.*

If $k \geq 2$, a $k$-chain twist is not equal to a single Dehn twist, so our theorem gives an infinite family of new non-geometric embeddings of the braid group in mapping class groups.

Our result, along with the other examples in the literature of non-geometric embeddings, raises the following question, which is an avenue of further research.

**Question 5.** *Is there a classification of all possible conjugacy classes of embeddings of the braid group in the mapping class group?*

*The Burau representation.* The Burau representation of the braid group, $\mathfrak{B} : B_n \to GL_n(\mathbb{Z}[t, t^{-1}])$, is an important representation with connections to knot theory. It is known that the Burau representation is faithful for $n = 2, 3$ [7], and not faithful for $n \geq 5$ [6, 24]. Whether or not the Burau representation is faithful for $n = 4$ remains an open question.

For each $k \geq 2$, the Birman-Hilden theorem applied to the Burau covers from Theorem 3 gives an embedding $B_n \hookrightarrow \mathrm{MCG}(\Sigma)$ for some surface $\Sigma$. Composing this embedding with the action on the first homology of $\Sigma$ gives a representation $\mathfrak{B}_k : B_n \to GL_n(\mathbb{Z}[t]/p_k)$ where $p_k = t^{k-1} + \cdots + t + 1$. In work in progress, Alan McLeay and I show the following connection between the $\mathfrak{B}_k$ and the Burau representation.

**Theorem 6** (Ghaswala-McLeay)**.** $\bigcap_{k=2}^{\infty} \ker(\mathfrak{B}_k) = \ker \mathfrak{B}$.

An explicit generating set for $\ker(\mathfrak{B}_2)$ was found in recent work of Brendle, Margalit, and Putman [14]. However, the groups $\ker(\mathfrak{B}_k)$ for $k \geq 3$ remain poorly understood, and are the focus of an ongoing research project with Alan McLeay.

**Project 7.** *Understand the groups* $\ker(\mathfrak{B}_k)$ *with an eye towards understanding the intersection* $\bigcap_{k=2}^{\infty} \ker(\mathfrak{B}_k)$.

The Burau representation can also be defined by taking an infinite cyclic cover of the disk punctured at $n$ points, and looking at the action of the braid group on the first homology of the covering space. The covering space is a surface of infinite-type, so studying the kernel of the Burau representation amounts to studying the subgroup of the mapping class group of an infinite-type surface that acts trivially on the first homology of the surface. This is also an avenue of attack being persued by Alan McLeay and myself.

1.3. **Subsurfaces and omnipresent arcs.** Finite-type surfaces have the property that they do not contain any proper homeomorphic subsurfaces, that is, finite-type surfaces do not contain smaller copies of themselves. In work with Federica Fanoni and Alan McLeay, we prove the following fundamental result about the topology of infinite-type surfaces, which intuitively says that all infinite-type surfaces exhibit self-similar topology in some sense, much like the self-similarity of fractals.

**Theorem 8** (Fanoni-Ghaswala-McLeay [18])**.** *A surface is of infinite-type if and only if it contains a homeomorphic subsurface.*

One of the main tools used in the study of mapping class groups of finite-type surfaces, is the action of the mapping class group on a particular graph called the *curve complex*. The curve complex is desirable because as a graph it has infinite diameter, and it is $\delta$-hyperbolic. It is tempting to use the action of big mapping class groups on the curve complex of an infinite-type surface to obtain results. Disappointingly, the curve complex for infinite-type surfaces has diameter 2.

However, not all hope is lost! Using Theorem 8, we construct a graph called the *omnipresent arc graph* on which the mapping class group of the infinite-type surface acts. If we restrict our attention to the family of stable surfaces, we have the following:

**Theorem 9** (Fanoni-Ghaswala-McLeay [18])**.** *For almost all stable surfaces, the omnipresent arc graph is infinite-diameter and $\delta$-hyperbolic.*

Until our result, all constructions of infinite-diameter and $\delta$-hyperbolic arc graphs in the literature required there to be at least one, but at most finitely many punctures on the surface. Our result covers a much larger family of surfaces.

The condition that the surface is stable is an intriguing one. Roughly, a surface is *stable* if once you go out far enough to infinity in any direction, the topology of the surrounding surface remains the same. Figure 1 gives an example of a stable surface, whereas Figure 3 is an unstable surface. There are many results in the literature which rely on the stability assumption, but it would be excellent to be able to remove these assumptions. Naturally this inspires the following project, which is just one of many basic open problems surrounding the topology of infinite-type surfaces.

**Project 10.** *Understand the topology of the endspace of unstable surfaces.*

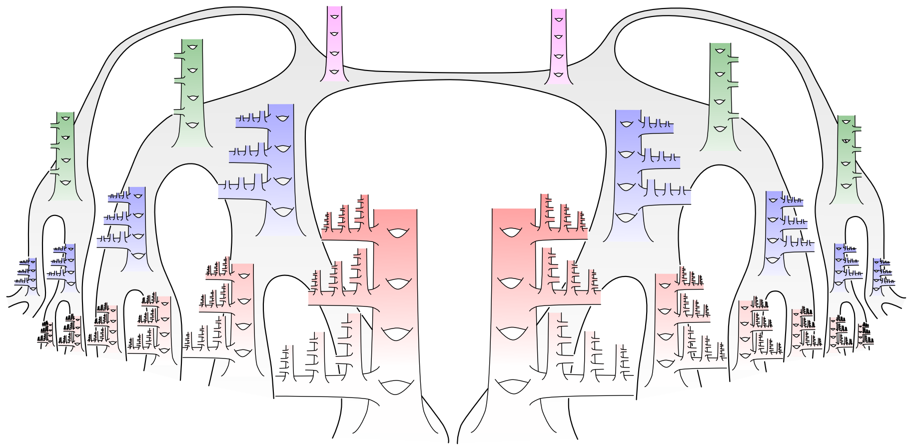

FIGURE 3. An unstable infinite-type surface

## 2. Left-orderable and circularly-orderable groups

A group is *left-orderable* (*circularly-orderable*) if there exists a total ordering (circular ordering) on the group that is invariant under left-multiplication. Intuitively, a left-orderable group is one which can be placed on a line so that left-multiplication by any element preserves the order of this line. The groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are all left-orderable (with the usual ordering), but so are many fundamental groups of 3-manifolds, braid groups, and free groups. A circularly-orderable group is one which can be placed around a circle so that left-multiplication by any element preserves the order of elements around the circle. For example, the groups $\mathbb{Z}/k\mathbb{Z}$, $\mathbb{Q}/\mathbb{Z}$, and $S^1$ are all circularly-orderable, as are mapping class groups of surfaces with a single puncture.

Although these may seem like somewhat contrived conditions, there are many connections to low-dimensional topology. Indeed, if $G$ is countable, then being left-orderable is equivalent to embedding in $\mathrm{Homeo}^+(\mathbb{R})$, and being circularly-orderable is equivalent to embedding in $\mathrm{Homeo}^+(S^1)$. Therefore these combinatorial conditions completely characterize groups that act on 1-manifolds! Additionally, the L-space conjecture predicts a connection between between taut foliations, Heegaard-Floer homology and left-orderability of the fundamental groups of 3-manifolds [13, 21]. There are also direct connections between the existence of a circular ordering on a 3-manifold and the topology of the manifold via Thurston's universal circle construction [15].

In this setting, in order to left-order or circularly-order the fundamental group of a 3-manifold, it is important to understand when you can left-order or circularly-order an amalgamated free product of left-ordered or circularly-ordered groups. Necessary and sufficient conditions to left-order an amalgamated free product are given in work of Budlov and Glass [11].

Adam Clay and I approach the problem in the circularly-ordered case by proving an equivalence of categories between appropriately defined categories of left-ordered and circularly ordered groups. Given a circularly-ordered group $G$, one is able to define a left-ordered group $\widetilde{G}$ which is a central extension of $G$. We are able to use our categorical viewpoint to prove the following theorem:

**Theorem 11** (Clay-Ghaswala [16])**.** *The amalgamated free product $*_A G_i$ of circularly-ordered groups is circularly orderable with an order extending that on the factors if and only if $*_{\widetilde{A}} \widetilde{G_i}$ is left orderable with an order extending that on the factors.*

### 2.1. The obstruction spectrum.

One effective strategy for showing a group is left-orderable is to first show it is circularly-orderable, usually by exhibiting an action on the circle $S^1$. Then, by a cohomological or dynamical argument conclude the action on $S^1$ must have a global fixed point, implying the group is left-orderable. It is natural therefore to study obstructions to left-orderability on circularly-orderable groups.

Left-orderable groups are torsion-free, whereas circularly-orderable groups can admit finite subgroups as long as they are cyclic. A question of Kathryn Mann asks if torsion is the only obstruction. More specifically, are there torsion-free circularly-orderable groups that are not left-orderable? In work with Jason Bell and Adam Clay, we answer this question by showing there exist torsion-free circularly-orderable groups that are not left-orderable. In the same paper, we prove a fundamental new characterization of left-orderability of a group.

**Theorem 12** (Bell-Clay-Ghaswala [5]). *A circularly-orderable group $G$ is left-orderable if and only if $G \times \mathbb{Z}/n\mathbb{Z}$ is circularly-orderable for all $n \geq 2$.*

Theorem 12 allows us to define the *obstruction spectrum* of a circularly-ordered group $G$ as the subset of natural numbers with the property that $G \times \mathbb{Z}/n\mathbb{Z}$ is not circularly-orderable. When $G$ has torsion, the obstruction spectrum detects the torsion. However, when $G$ is torsion-free but not left-orderable, the obstruction spectrum is detecting something more mysterious.

In a follow up paper, Adam Clay and I show that the obstruction spectrum detects cohomological information of the circular orderings on a group. Given a circular ordering $c$ on a group $G$, $c$ represents a cohomology class $[c] \in H^2(G; \mathbb{Z})$.

**Theorem 13** (Clay-Ghaswala [17]). *Let $G$ be a circularly-orderable group. The group $G \times \mathbb{Z}/n\mathbb{Z}$ is circularly orderable if and only if there exists a circular ordering $c$ on $G$ such that $[c]$ is divisible by $n$.*

We use Theorem 13 to compute the obstruction spectrum for several 3-manifold groups, as well as some mapping class groups (see Section 3).

## 3. Orderability and mapping class groups

Mapping class groups of finite-type surfaces give many examples of left-orderable and circularly-orderable groups. It turns out that if the surface has one puncture, then its mapping class group is circularly-orderable, and if the surface has non-empty boundary, then its mapping class group is left-orderable.

Using Theorem 13, Adam Clay and I proved the following result, which shows the obstruction spectrum of the mapping class group of a once-punctured surface is as big as it can be.

**Theorem 14** (Clay-Ghaswala [17]). *The obstruction spectrum of the mapping class group of a finite-type surface with one puncture is $\mathbb{N}_{\geq 2}$.*

Intriguingly, since the order of torsion elements in the mapping class group is bounded, the theorem implies that torsion is not the only obstruction to left-orderability. This result is only for finite-type mapping class groups, but I conjecture that Theorem 14 holds for big mapping class groups as well.

For finite-type surfaces without boundary, the mapping class group is not left-orderable since it contains torsion. However, it is known that mapping class groups contain finite index subgroups that are torsion free (namely the congruence subgroups).

**Question 15.** *Are the congruence subgroups of the mapping class group left-orderable? More generally, is the mapping class group virtually left-orderable?*

Given a left-orderable group $G$, the space of left orderings $LO(G)$ can be appropriately topologized to become a compact Hausdorff space, and it is naturally a subspace of the space of all circular orderings $CO(G)$ on $G$. Currently little is known about the space of left orders on a mapping class group.

**Project 16.** *Understand the space of left-orders $LO(\mathrm{MCG}(\Sigma))$ where $\Sigma$ is a surface with boundary.*

As part of Project 16, in work in progress Adam Clay and I show that for the mapping class group of a surface $\Sigma$ with one boundary component, the Dehn twist about the boundary component is a special element when looking at left-orderings.

**Theorem 17** (Clay-Ghaswala). *The Dehn twist about the boundary component is cofinal in every left ordering of the mapping class group.*

Combining this with recent work of Idrissa Ba and Adam Clay [4], we are able to conclude that every left ordering of $\mathrm{MCG}(\Sigma)$ is an accumulation point of genuine circular orderings (a circular ordering that is not a left-ordering).

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